

REPORT DOCUMENTATION PAGE			<i>Form Approved</i> OMB No. 0704-0188	
<small>Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden to Washington Headquarters Service, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188) Washington, DC 20503.</small>				
PLEASE DO NOT RETURN YOUR FORM TO THE ABOVE ADDRESS.				
1. REPORT DATE (DD-MM-YYYY) 06-01-2017		2. REPORT TYPE Technical Report		3. DATES COVERED (From - To)
4. TITLE AND SUBTITLE Shot Group Statistics for Small Arms Applications		5a. CONTRACT NUMBER N/A		
		5b. GRANT NUMBER N/A		
		5c. PROGRAM ELEMENT NUMBER N/A		
6. AUTHOR(S) W. J. Gammon, Ph.D.		5d. PROJECT NUMBER N/A		
		5e. TASK NUMBER N/A		
		5f. WORK UNIT NUMBER N/A		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Product Manager Infantry Weapons Program Manager Infantry Weapon Systems Marine Corps Systems Command Quantico, VA 22134			8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) Product Manager Infantry Weapons Program Manager Infantry Weapon Systems Marine Corps Systems Command Quantico, VA 22134			10. SPONSOR/MONITOR'S ACRONYM(S) MARCORSYSCOM	
			11. SPONSORING/MONITORING AGENCY REPORT NUMBER	
12. DISTRIBUTION AVAILABILITY STATEMENT Approved for public release; distribution is unlimited.				
13. SUPPLEMENTARY NOTES				
14. ABSTRACT <p>This report provides a systematic analysis of dispersion measures of shot groups from firearms. A dispersion measure is a random variable, and if its probability distribution is known with sufficient accuracy, then it can be used to make a sound statistical inference on the unknown, population standard deviations of the x and y impact-point positions. The dispersion measures treated in this report are the univariate measures: range, mean deviation, and standard deviation; and the bivariate measures: extreme spread, mean radius, and radial standard deviation. Analysis is presented as applied to one, n-round shot group and then is extended to treat multiple, n-round shot groups. A dispersion measure for multiple, n-round shot groups can be constructed by selecting one of the dispersion measures listed above, measuring the dispersion of each group, and averaging the dispersion over groups. This procedure was followed except for the construction of dispersion measures based on the standard deviation and radial standard deviation of multiple n-round shot groups, for which we used the root mean square of the dispersion measure over groups. For each dispersion measure, tables of means, standard deviations, and percentiles were computed for single, n-round shot groups (n = 2-30), as well as for treatments of various k multiples (k = 2-5, 10) of n-round shot groups (n = 2-5, 10, 25). The percentile tables provided in this report are to a considerable extent unavailable elsewhere, and their inclusion is of practical interest to those developing requirements or tests on firearms precision.</p>				
15. SUBJECT TERMS shot group statistics, dispersion, precision, percentile tables, range, extreme spread, mean radius, mean deviation, standard deviation, radial standard deviation, ballistics, firearms, small arms, marksmanship				
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT SAR	18. NUMBER OF PAGES 74
a. REPORT Unclassified	b. ABSTRACT Unclassified	c. THIS PAGE Unclassified	19a. NAME OF RESPONSIBLE PERSON W. J. Gammon, Ph.D.	
			19b. TELEPHONE NUMBER (Include area code) 703-784-3759	

Shot Group Statistics for Small Arms Applications

6/1/2017

Marine Corps Systems Command
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SHOT GROUP STATISTICS FOR SMALL ARMS APPLICATIONS

Abstract

This report provides a systematic analysis of dispersion measures of shot groups from firearms. A dispersion measure is a random variable, and if its probability distribution is known with sufficient accuracy, then it can be used to make a sound statistical inference on the unknown, population standard deviations of the x and y impact-point positions. The dispersion measures treated in this report are the univariate measures: range, mean deviation, and standard deviation; and the bivariate measures: extreme spread, mean radius, and radial standard deviation. Analysis is presented as applied to one, n -round shot group and then is extended to treat multiple, n -round shot groups. A dispersion measure for multiple, n -round shot groups can be constructed by selecting one of the dispersion measures listed above, measuring the dispersion of each group, and averaging the dispersion over groups. This procedure was followed except for the construction of dispersion measures based on the standard deviation and radial standard deviation of multiple n -round shot groups, for which we used the root mean square of the dispersion measure over groups. For each dispersion measure, tables of means, standard deviations, and percentiles were computed for single, n -round shot groups ($n = 2-30$), as well as for treatments of various k multiples ($k = 2-5, 10$) of n -round shot groups ($n = 2-5, 10, 25$). The percentile tables provided in this report are to a considerable extent unavailable elsewhere, and their inclusion is of practical interest to those developing requirements or tests on firearms precision.

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1. Introduction and Summary

This report provides a systematic analysis of dispersion measures of shot groups from firearms and was written to assist those developing requirements or tests on firearms precision. A shot group is a two-dimensional pattern generated by the impact points of the shots, which pierce a plane at some distance away from the firing line. For our purposes, this plane will be considered to be parallel to the firing line. As such, an n -round shot group consists of the impact points of the n shots, which have positions x_i and y_i along the horizontal and vertical directions, respectively. As an example, a 5-round shot group at 300 yards from an M40A5 sniper rifle is displayed in Figure 1, and its x and y impact points are listed in Table 1. The scatter observed in the shot group is due to the random nature of ballistic dispersion.

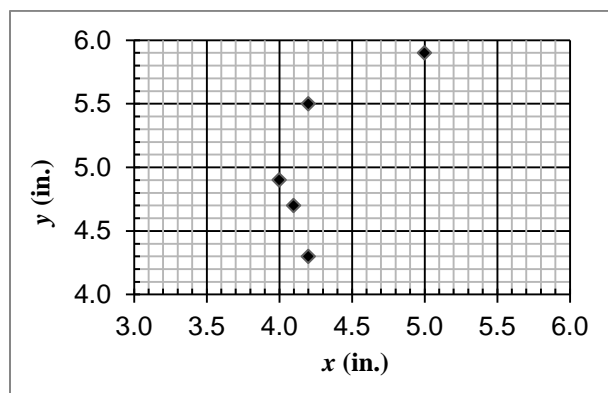


Figure 1. A 5-round shot group at 300 yards from an M40A5 sniper rifle fired from a test fixture.

Table 1. The impact points of the 5-round shot group displayed in Figure 1.

Shot Number	$x(\text{in.})$	$y(\text{in.})$
1	4.0	4.9
2	4.2	4.3
3	4.1	4.7
4	4.2	5.5
5	5.0	5.9

In the study of firearms precision, knowledge of the underlying, population standard deviations of the x and y impact-point positions is necessary, and dispersion measures are valuable tools to gain this knowledge. A dispersion measure is a random variable, and if its probability distribution is known with sufficient accuracy, then it can be used to make a sound statistical inference on the unknown, population standard deviations of the x and y impact-point positions. Throughout this report, it is assumed that the random variables x and y are normally distributed, and that the standard deviations of x and y are the same ($\sigma = \sigma_x = \sigma_y$). The dispersion measures treated in this report are the univariate measures (Section 3.1):

Range (R), Mean Deviation (MD), and Standard Deviation (SD); and the bivariate measures (Section 3.2): Extreme Spread (ES), Mean Radius (MR), and Radial Standard Deviation (RSD). Analysis of each dispersion measure is first presented as applied to one, n -round shot group and then is extended to treat multiple, n -round shot groups. Our analytical approach is described below.

First, an analysis of each dispersion measure is given as applied to one, n -round shot group ($n = 2-30$). For each dispersion measure, percentiles and the first two moments were computed in units of the population standard deviation, σ . Percentiles of these dispersion measures are of principal interest for small sample sizes, especially the upper and lower 10th percentiles. For large sample sizes, these probability distributions become nearly normally distributed. The probability distributions of SD and RSD are well known: $\sqrt{n}SD/\sigma$ and $\sqrt{n}RSD/\sigma$ are chi random variables with $(n - 1)$ and $2(n - 1)$ Degrees of Freedom (DoF), respectively. (For $\text{DoF} \geq 30$, chi distributions are nearly normally distributed.) The percentiles of SD and RSD are obtained by taking the square root of the percentiles of the corresponding chi-square distribution, and are thus easily computed—we used the built-in functions in Microsoft Excel 2010. Similarly for $n = 2$, the probability distributions of R , MD , ES , and MR are scaled chi random variables with one DoF (univariate) or two DoF (bivariate) and were computed in a similar fashion. Conversely for $n > 2$, the probability distributions of R , MD , ES , and MR lack convenient functional forms, and in this case, Monte Carlo simulations were used to estimate their percentiles. In addition, we provide analytical formulas for the means and variances of these dispersion measures treating one shot group, with exception to R and ES with $n > 2$ for which the moments are Monte Carlo estimates. Details on the Monte Carlo simulations used in this work are found in Section 2.

Our analysis was then extended to treat various k multiples ($k = 2-5, 10$) of n -round shot groups ($n = 2-30$) where the multiples are independent, repeat firings of n -round groups from the same population. Repeated firings of groups allow for a more precise estimate of σ . A dispersion measure for k , n -round shot groups can be constructed by selecting a dispersion measure, measuring the dispersion of each of the k groups, and averaging the dispersion over groups. This procedure was used to construct a dispersion measure for multiple, n -round shot groups based on R , MD , ES , and MR , which when averaged over the sampling of groups, yields the corresponding dispersion measures: Average R (AR), Average MD (AMD), Average ES (AES), and Average MR (AMR), respectively. Similarly, such a dispersion measure based on SD or RSD for multiple groups may be constructed; however, an alternative procedure was selected for convenience. Instead of averaging the SD (RSD) over groups, our measure of dispersion for multiple groups is the Root Mean Square (RMS) of the SD 's (RSD 's) over groups, which is denoted as $RMS-SD$ ($RMS-RSD$). The construction of $RMS-SD$ and $RMS-RSD$ was chosen, because their probability distributions are well known: $\sqrt{kn}RMS-SD/\sigma$ and $\sqrt{kn}RMS-RSD/\sigma$ are chi random variables with $k(n-1)$ and $2k(n-1)$ DoF, respectively.

Whereas, in general, the probability distributions of AR , AMD , AES , and AMR lack convenient functional forms; therefore, Monte Carlo simulations were used to estimate their percentiles. From the central limit theorem, the probability distribution of the average over a dispersion measure is asymptotically normal as the number of groups averaged over increases. Our Monte Carlo estimates of percentiles can be used to determine the error associated with approximating the probability distribution with a normal one. The largest errors committed in using this approximation occur in the tails of the distribution. Convergence to a normal distribution is quick, justifying the normal-distribution approximation for many cases; therefore, practical interest in the percentile tables of AR , AMD , AES , and AMR will typically be restricted to cases involving a few groups with small sample size. For $n \geq 10$ and $k > 1$, the error associated with this approximation is no more than 10% for any of the percentiles computed in this work. Finally, supplementing our results on percentiles, the first two moments were computed for the dispersion

measures treating multiple shot groups. Wrapping up the description of our analytical approach, we present Table 2, which provides a listing of the dispersion measures examined in this report.

Table 2. Dispersion measures examined in this report.

Dispersion Measures			
Univariate		Bivariate	
One Shot Group	Multiple Shot Groups	One Shot Group	Multiple Shot Groups
<i>R</i>	<i>AR</i>	<i>ES</i>	<i>AES</i>
<i>MD</i>	<i>AMD</i>	<i>MR</i>	<i>AMR</i>
<i>SD</i>	<i>RMS-SD</i>	<i>RSD</i>	<i>RMS-RSD</i>

Next, a review of the literature is provided as pertaining to the dispersion measures covered in this report. In Grubbs' monograph,¹ he provided the means, standard deviations, and 95th percentiles of *R*, *MD*, *SD*, *ES*, *MR*, and *RSD* for single, *n*-round shot groups (*n* = 2-20). Taylor and Grubbs² also provided percentile tables of *ES* along with the first four moments (*n* = 3-10, 15, 20, 25, 28, 30, 31, 34). In addition, Pearson and Hartley³ published tables of the upper and lower 10th percentiles of *R* (*n* = 2-20) and *MD* (*n* = 2-10) as well as the means and standard deviations of *R* and *MD* for various sample sizes (*n* = 2-20, 30, 60). The results presented in this work extend upon those reported in previous studies. To the author's knowledge, published percentile tables of *MR* are unavailable, and a comprehensive treatment of multiple shot groups is unavailable. The goal of this report is to fill this gap and provide a thorough treatment of dispersion measures of shot groups in a single source.

In summary, for each dispersion measure treated herein, tables of means, standard deviations, and percentiles are provided for single, *n*-round shot groups (*n* = 2-30), as well as for treatments of various *k* multiples (*k* = 2-5, 10) of *n*-round shot groups (*n* = 2-5, 10, 25), where the multiples are independent, repeat firings from the same population. When possible, a sketch of the derivation of analytical formulas for the first two moments is provided. This approach is perhaps a theoretical distraction to many, but to the interested reader, it provides the method to compute moments for (*n*, *k*) combinations not covered in this report. To facilitate use of these results, numerous examples are provided, which will prove indispensable to those developing requirements or tests on firearms precision.

2. Monte Carlo Simulations

Monte Carlo simulations were used to compute percentiles of the probability distributions of *R*, *MD*, *ES*, and *MR* for single, *n*-round shot groups, as well as the percentiles of Average *R*, Average *MD*, Average *ES*, and Average *MR* for *k* multiples of *n*-round shot groups. Monte Carlo simulations were also used to estimate the means and variances of *R*, *ES*, *AR*, and *AES*, and their Monte Carlo estimates are reported in units of the population standard deviation, σ . Simulated moments and percentiles are based on *N* Monte Carlo samples (here, *N* = 20,000) of each *n* impact-point position in each of the *k* sampled, *n*-round shot groups. Thus, *knN* and *2knN* random samples were generated for one Monte Carlo run to simulate moments and percentiles of the univariate and bivariate distributions, respectively. All Monte Carlo simulations were repeated for a total of 9 runs, and Monte Carlo estimates of the simulated moments and percentiles are reported as the sample mean (\bar{Z}) over the replications (*r* = 9). In addition, we report the Monte Carlo Error (*MCE*) associated with each Monte Carlo estimate, which provides a measure of the

variability between simulations. The *MCE* is defined as the sample standard deviation of the Monte Carlo estimates,⁴ and the $1-2\alpha$ confidence interval of the population mean can be estimated by:

$$\left(\bar{Z} - \frac{t_{1-\alpha}^{MCE}}{\sqrt{r}}, \bar{Z} + \frac{t_{1-\alpha}^{MCE}}{\sqrt{r}} \right),$$

where $t_{1-\alpha}$ is the $100(1-\alpha)$ percentile of the Student's *t*-distribution with $(r-1)$ DoF.

Random samples of the dispersion measures were constructed by sampling the normal distributions of the x and y impact-point positions. Microsoft Excel 2010 was used for all Monte Carlo simulations, and random samples from a normal distribution were generated by the inverse transform technique using the `NORM.INV(RAND(),0,1)` function, which returns a random sample from a normal distribution with zero mean and standard deviation of one. These random samples were used to construct N samples of the dispersion measure, whose percentiles were computed using the `PERCENTILE` function.

3. Results and Discussion

3.1. Univariate Measures of Dispersion

3.1.1. Range, One Shot Group

The univariate measure of dispersion known as the Range (R) is given by:

$$R = (x_n - x_1), \text{ here } x_1 \leq x_2 \leq x_3 \cdots \leq x_n. \quad (1)$$

R is measured by projecting the n shot positions on the x -axis and then measuring the distance between the two furthest shots along that projection. If the x -components are horizontal distances, then R is referred to as the Extreme Horizontal Dispersion (*EHD*), and if these are vertical distances, then it is referred to as the Extreme Vertical Dispersion (*EVD*).¹ For convenience, recast (1) as:

$$R = \max_{i,j} \left\{ \sqrt{(x_i - x_j)^2} \right\}; \quad (2)$$

that is, R is the maximum of the distances between all possible pairs of x_i and x_j . Dividing both sides by σ :

$$R/\sigma = \max_{i,j} \left\{ \sqrt{((x_i - x_j)/\sigma)^2} \right\}, \quad (3)$$

where σ is the population standard deviation of the x -components of impact points. Next, consider the special case of (3) with $n = 2$:

$$R/\sigma = |(x_2 - x_1)/\sigma|$$

$$R/\sigma = \sqrt{2} |(x_2 - x_1)/\sqrt{2}\sigma|.$$

Since the x_i are normally distributed and independent, $(x_2 - x_1)/\sqrt{2}\sigma$ is normally distributed with zero mean and unity variance; therefore,

$$R/\sigma = \sqrt{2}\chi(1) \quad (n = 2), \quad (4)$$

where $\chi(1)$ is a chi-distributed random variable with one degree of freedom. Continuing with $n = 2$, the mean of R/σ is given by:

$$\langle R/\sigma \rangle = \sqrt{2}\langle \chi(1) \rangle$$

$$\langle R/\sigma \rangle = 2/\Gamma(\frac{1}{2}),$$

where Γ is the gamma function, and inserting the value of $\Gamma(\frac{1}{2})$ gives:

$$\langle R/\sigma \rangle = 2/\sqrt{\pi} \quad (n = 2), \quad (5)$$

and its variance is computed from:

$$\begin{aligned} \text{Var}(R/\sigma) &= \langle (R/\sigma)^2 \rangle - \langle R/\sigma \rangle^2 \\ &= 2[\langle \chi^2(1) \rangle - 2/\pi] \end{aligned}$$

$$\text{Var}(R/\sigma) = 2(1 - 2/\pi) \quad (n = 2). \quad (6)$$

Taking the square root of (6) gives the standard deviation:

$$\text{Std}(R/\sigma) = \sqrt{2(1 - 2/\pi)} \quad (n = 2). \quad (7)$$

For $n = 2$, simple analytical results are provided for the probability distribution and first two moments of R/σ . If $n > 2$, an extension of these analytical results is unavailable, because the exact probability distributions are unknown. As such, computation of the percentiles and moments would require methods such as numerical integration techniques³ or Monte Carlo. For $n > 2$, Monte Carlo simulations were chosen to compute the percentiles and moments, because these methods are easy to implement and obtain results with sufficient accuracy for practical firearms applications. This approach was taken throughout this report for other dispersion measures with probability distributions not easily expressed in a convenient form.

In Table 3, results are presented for the means, standard deviations, and 5th and 95th percentiles of R/σ for sample sizes of $n = 2-30$. For $n = 2$, the means and standard deviations displayed in the table are from (5) and (7), and the percentiles were determined from those of a chi-distributed random variable (one degree of freedom) scaled by $\sqrt{2}$ as shown in (4). For $n > 2$, all tabulated results are Monte Carlo estimates. A listing of percentiles of R/σ is found in Appendix 1-1, along with the *MCE*'s of the estimated percentiles.

In earlier work, Pearson and Hartley³ published means and standard deviations of R/σ ($n = 2-20, 30, 60$), and the means and standard deviations in Table 3 agree with their results within 0.5%. Pearson and Hartley³ also published values of the upper and lower 10th percentiles of R/σ ($n = 2-20$) to two decimal places. Comparison of their results with the percentiles in Appendix 1-1 is generally in very good agreement. However, we must remark on the 99.9 percentiles listed in Appendix 1-1. We note that as the sample size increases, these percentiles are expected to monotonically increase and eventually reach an asymptotic value. Up to $n = 14$, these percentiles monotonically increase; however, as the sample size further increases, small fluctuations disrupting this behavior are observed. This is perhaps not surprising, since only 20 samples per Monte Carlo replication are expected above the 99.9 percentile (see Section 2).

More Monte Carlo replications would be required to smooth out this noisy behavior to attain better accuracy. This effort was not pursued—we only make a cautionary note, which indicates caution should be taken when estimating a confidence interval around the Monte Carlo estimates of these percentiles as prescribed in Section 2. Similar behavior may be observed in other results in this report, but any such behavior is less pronounced and will not be considered further.

Table 3. Table of means, standard deviations, and 5th and 95th percentiles of R/σ for sample sizes of $n = 2$ -30. Numbers reported in parentheses are the *MCE*'s of the value of the mean or standard deviation preceding it.

n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
2	1.1284	0.8525	0.7555	0.0887	2.7718	0.0786	2.4565
3	1.695 (0.004)	0.890 (0.004)	0.525	0.431	3.317	0.254	1.957
4	2.060 (0.006)	0.879 (0.006)	0.426	0.761	3.625	0.369	1.759
5	2.325 (0.007)	0.863 (0.005)	0.371	1.032	3.852	0.444	1.657
6	2.534 (0.003)	0.848 (0.004)	0.335	1.253	4.032	0.495	1.591
7	2.704 (0.006)	0.833 (0.005)	0.308	1.443	4.172	0.534	1.543
8	2.848 (0.006)	0.818 (0.003)	0.287	1.602	4.286	0.563	1.505
9	2.971 (0.007)	0.808 (0.005)	0.272	1.739	4.384	0.585	1.476
10	3.079 (0.005)	0.796 (0.003)	0.259	1.866	4.475	0.606	1.453
11	3.175 (0.003)	0.788 (0.004)	0.248	1.975	4.554	0.622	1.434
12	3.260 (0.005)	0.780 (0.006)	0.239	2.069	4.624	0.635	1.419
13	3.335 (0.005)	0.772 (0.003)	0.231	2.164	4.686	0.649	1.405
14	3.407 (0.005)	0.763 (0.004)	0.224	2.241	4.742	0.658	1.392
15	3.473 (0.006)	0.756 (0.003)	0.218	2.323	4.795	0.669	1.381
16	3.531 (0.005)	0.750 (0.006)	0.212	2.386	4.845	0.676	1.372
17	3.589 (0.004)	0.747 (0.005)	0.208	2.450	4.898	0.683	1.365
18	3.642 (0.005)	0.739 (0.004)	0.203	2.517	4.939	0.691	1.356
19	3.688 (0.006)	0.733 (0.003)	0.199	2.571	4.971	0.697	1.348
20	3.734 (0.005)	0.729 (0.003)	0.195	2.622	5.015	0.702	1.343
21	3.776 (0.004)	0.727 (0.003)	0.193	2.669	5.048	0.707	1.337
22	3.822 (0.003)	0.722 (0.003)	0.189	2.724	5.088	0.713	1.331
23	3.859 (0.003)	0.716 (0.003)	0.186	2.769	5.107	0.717	1.323
24	3.895 (0.007)	0.711 (0.003)	0.183	2.813	5.143	0.722	1.320
25	3.928 (0.005)	0.710 (0.006)	0.181	2.853	5.179	0.726	1.319
26	3.963 (0.006)	0.707 (0.004)	0.178	2.887	5.202	0.728	1.313
27	3.999 (0.004)	0.702 (0.003)	0.176	2.932	5.228	0.733	1.307
28	4.026 (0.005)	0.700 (0.003)	0.174	2.964	5.254	0.736	1.305
29	4.058 (0.006)	0.695 (0.004)	0.171	3.002	5.275	0.740	1.300
30	4.085 (0.005)	0.692 (0.004)	0.169	3.035	5.300	0.743	1.297

Notes Table 3: The first column displays the number of shots (n) in the group. The second and third columns display the means and standard deviations. The fourth column displays the Coefficient of Variation (*CV*), which is computed by dividing the standard deviation (third column) by the mean (second column). The *CV* (see (9)) is the standard deviation of the ratio of the unbiased estimate of the population standard deviation ($\hat{\sigma}$) to σ (i.e., $\hat{\sigma}/\sigma$). The fifth and sixth columns display the 5th and 95th percentiles, respectively. The seventh and eighth columns show the 5% and 95% Confidence Levels (CL) of the ratio of $\hat{\sigma}/\sigma$ (see (17)), respectively, and values in these columns were computed by dividing the 5th and 95th percentiles by the corresponding mean. Taken together, the values in the seventh and eighth columns provide the 90% confidence interval of $\hat{\sigma}/\sigma$. Values in the table reported to 3 decimal places indicate a result based on Monte Carlo estimates.

Example 1. Compute an unbiased estimate of σ using the Extreme Vertical Dispersion of the 5-round shot group in Table 1.

Using the values in Table 1, $EVD = 1.6''$ as computed from (2). An unbiased estimate of the population standard deviation ($\hat{\sigma}$) is computed from:

$$\hat{\sigma} = M / \langle M / \sigma \rangle, \quad (8)$$

where M is the dispersion measure—in this case $M = R$. (Computing the mean of (8) demonstrates $\hat{\sigma}$ is an unbiased estimate.) Substituting the computed EVD and mean of R/σ for one, 5-round group (Table 3):

$$\hat{\sigma} = 1.6'' / 2.325 = 0.69''.$$

Example 2. Select the Range as the dispersion measure, and determine the Coefficient of Variation for a 5-round group.

The Coefficient of Variation is defined by:

$$CV \equiv Std(\hat{\sigma} / \sigma).$$

Using (8) rewrite the above:

$$CV = Std(M / \sigma) / \langle M / \sigma \rangle, \quad (9)$$

and consulting Table 3, the CV for a 5-round group is:

$$CV = 0.371.$$

Note multiplying CV by σ gives the standard deviation of the unbiased estimate $\hat{\sigma}$.

Example 3. Using the Extreme Vertical Dispersion, find the 90% Confidence Interval (CI) of the population standard deviation of the 5-round group listed in Table 1.

Write the probability distribution of the dispersion measure M in units of the population standard deviation:

$$T \equiv M / \sigma. \quad (10)$$

The 90% CI of T is:

$$T_{0.05} \leq M / \sigma \leq T_{0.95}, \quad (11)$$

from which the 90% CI of σ is determined:

$$M / T_{0.95} \leq \sigma \leq M / T_{0.05} \quad (90\% \text{ CI}), \quad (12)$$

where $M = EVD$. Inserting the EVD from Example 1 with the needed percentiles for a 5-round group (Table 3):

$$1.6'' / 3.852 \leq \sigma \leq 1.6'' / 1.032$$

$$0.42'' \leq \sigma \leq 1.55'' \quad (90\% \text{ CI}).$$

Example 4. Repeat the calculation in Example 3 for a 95% Confidence Interval of the population standard deviation.

Referring to Appendix 1-1, the 97.5 and 2.5 percentiles of R/σ for a 5-round shot group are 4.190 and 0.849, respectively. Analogous to (12):

$$M / T_{0.975} \leq \sigma \leq M / T_{0.025} \quad (95\% \text{ CI}).$$

Computation gives:

$$1.6''/4.190 \leq \sigma \leq 1.6''/0.849$$

$$0.38'' \leq \sigma \leq 1.88'' \quad (95\% \text{ CI}).$$

3.1.2. Average Range, Multiple Shot Groups

For multiple, n -round shot groups, the range is averaged over all groups to provide a measure of dispersion:

$$AR/\sigma = \frac{1}{k} \sum_{i=1}^k R_i/\sigma, \quad (13)$$

where AR is the Average Range, R_i is the range of the i^{th} n -round shot group, and k is the total number of groups. The mean of (13) is:

$$\langle AR/\sigma \rangle = \langle R/\sigma \rangle, \quad (14)$$

since the mean of R for one, n -round shot group is the same as the mean of an average over multiple, n -round groups. The standard deviation of AR/σ is:

$$Std(AR/\sigma) = \frac{1}{\sqrt{k}} Std(R/\sigma), \quad (15)$$

as expected for an average over k groups. The Coefficient of Variation of AR is defined by its standard deviation (15) divided by its mean (14):

$$CV(AR) = \frac{1}{\sqrt{k}} CV(R), \quad (16)$$

which indicates that its CV is the same as the CV of R divided by \sqrt{k} .

In general the probability distributions of AR/σ do not have handy functional forms for easy computations. However, we note that for $k = 2$ and $n = 2$, AR/σ is the average of the sum of two scaled chi random variables each with one DoF (see (4)). The joint density function of the sum of two independent random variables is given by the convolution of their probability density functions, and the joint density function can be integrated to determine its percentiles. This is a fairly easy exercise for $k = 2$ and $n = 2$, but is laborious for larger values of k . As such, Monte Carlo simulations were selected to compute all percentiles of AR/σ . Similarly, this remark applies to the percentile computations of AMD/σ , AES/σ , and AMR/σ which are presented in later sections.

In Table 4, results are presented for the means, standard deviations, and 5th and 95th percentiles of AR/σ for n -round shot groups ($n = 2, 5, 10, 25$) fired for a total of k groups ($k = 1, 5, 10$ for each of the listed n). For $n = 2$, the listed mean is $2/\sqrt{\pi}$ via (5), and the standard deviations were computed from (7) divided by \sqrt{k} . For $n = 2$ and $k > 1$, the percentiles are Monte Carlo estimates. For $n > 2$, all tabulated results are from Monte Carlo simulations, and estimates of the means are from simulations with $k = 10$. A listing of the percentiles of AR/σ for the various n and k is found in Appendix 1-2, along with the MCE 's of the estimated percentiles.

We know from the central limit theorem that as k increases, AR/σ is asymptotically normal. The largest errors committed in using this approximation are, of course, in the tails of the distribution. If k is large enough, AR/σ is well approximated by a normal distribution with mean $\langle R/\sigma \rangle$ and standard deviation $Std(R/\sigma)/\sqrt{k}$. A similar argument obviously applies to AMD/σ , AES/σ , and AMR/σ (Sections 3.1.4,

3.2.2 and 3.2.4, respectively). The Monte Carlo estimates of percentiles given in Appendix 1-2 allow the error to be determined when making this approximation to AR/σ . As an example, consider the expected mean and standard deviation of the Range for one, 5-round group, which from Table 3 are 2.325 and 0.863, respectively. The normal approximation to AR/σ for 3, 5-round shot groups would be a normal distribution with a mean of 2.325 and standard deviation of $0.863/\sqrt{3}$. The 5th percentile of this normal distribution is 1.505, which can be compared to 1.545 (the corresponding 5th percentile reported in Table 4). For this example, the error is only 2.6%. As demonstrated, our results can be used to quickly determine the error in assuming this approximation. Error within a few percent is suitable for most firearms applications, and primary concerns with using this approximation are for dispersion measures involving averages over a few groups with a small sample size per group.

Table 4. Table of means, standard deviations, and 5th and 95th percentiles of AR/σ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$). Numbers reported in parentheses are the *MCE*'s of the value of the mean or standard deviation preceding it.

k	n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
1	2	1.1284	0.8525	0.7555	0.0887	2.7718	0.0786	2.4565
2			0.6028	0.5342	0.286	2.239	0.253	1.984
3			0.4922	0.4362	0.411	2.015	0.364	1.786
4			0.4263	0.3778	0.494	1.892	0.437	1.676
5			0.3813	0.3379	0.554	1.804	0.491	1.599
10			0.2696	0.2389	0.709	1.595	0.629	1.414
1	3	1.693 (0.002)	0.890 (0.004)	0.526	0.431	3.317	0.255	1.960
2			0.628 (0.005)	0.371	0.754	2.809	0.445	1.660
3			0.512 (0.003)	0.302	0.907	2.586	0.536	1.527
4			0.444 (0.0016)	0.262	1.004	2.464	0.593	1.456
5			0.396 (0.003)	0.234	1.073	2.374	0.634	1.402
10			0.281 (0.0013)	0.166	1.249	2.171	0.738	1.282
1	4	2.058 (0.002)	0.879 (0.006)	0.427	0.761	3.625	0.370	1.761
2			0.622 (0.003)	0.302	1.109	3.146	0.539	1.528
3			0.508 (0.003)	0.247	1.269	2.939	0.617	1.428
4			0.439 (0.002)	0.213	1.369	2.809	0.665	1.365
5			0.393 (0.002)	0.191	1.442	2.734	0.701	1.328
10			0.278 (0.0009)	0.135	1.614	2.529	0.784	1.229
1	5	2.326 (0.0015)	0.863 (0.005)	0.371	1.032	3.852	0.444	1.656
2			0.612 (0.003)	0.263	1.379	3.384	0.593	1.455
3			0.499 (0.003)	0.214	1.545	3.184	0.664	1.369
4			0.434 (0.002)	0.186	1.642	3.069	0.706	1.319
5			0.385 (0.0009)	0.166	1.715	2.982	0.737	1.282
10			0.273 (0.0016)	0.117	1.890	2.787	0.812	1.198
1	10	3.077 (0.0013)	0.796 (0.003)	0.259	1.866	4.475	0.606	1.454
2			0.562 (0.004)	0.183	2.199	4.042	0.714	1.314
3			0.461 (0.0019)	0.150	2.349	3.869	0.763	1.257
4			0.398 (0.003)	0.129	2.444	3.754	0.794	1.220
5			0.357 (0.003)	0.116	2.509	3.681	0.815	1.196
10			0.252 (0.002)	0.082	2.672	3.499	0.868	1.137
1	25	3.931 (0.0015)	0.710 (0.006)	0.181	2.853	5.179	0.726	1.318
2			0.500 (0.002)	0.127	3.150	4.793	0.801	1.219
3			0.409 (0.003)	0.104	3.285	4.628	0.836	1.177
4			0.354 (0.005)	0.090	3.369	4.531	0.857	1.153
5			0.316 (0.0016)	0.080	3.429	4.467	0.872	1.136
10			0.225 (0.0012)	0.057	3.571	4.309	0.908	1.096

Notes Table 4: The first and second columns display the number of groups (k) and the number of shots (n) in the group, respectively. The third and fourth columns display the means and standard deviations. The fifth column displays the Coefficient of Variation (CV), which was computed by dividing the standard deviation (fourth column) by the mean (third column). The CV (see (9)) is the standard deviation of the ratio of the unbiased estimate of the population standard deviation ($\hat{\sigma}$) to σ (i.e., $\hat{\sigma}/\sigma$). The sixth and seventh columns provide the 5th and 95th percentiles, respectively. The eighth and ninth columns provide the 5% and 95% Confidence Levels (CL) of the ratio of $\hat{\sigma}/\sigma$ (see (17)), respectively, and these values were computed by dividing the 5th and 95th percentiles by the corresponding mean. Taken together the values in the eighth and ninth columns provide the 90% confidence interval of $\hat{\sigma}/\sigma$. Values in the table reported to 3 decimal places indicate a result based on Monte Carlo estimates.

Example 5. If the Average Range is used as the dispersion measure, how many 3-round shot groups need to be fired to get at least the same Coefficient of Variation as from firing 5, 10-round shot groups?

Equating CV 's gives:

$$CV(AR(k, 3)) = CV(AR(5, 10)),$$

where $AR(k, n)$ denotes the average range for k, n -round shot groups. Using (16), this is rewritten as:

$$CV(R(3))/\sqrt{k} = CV(R(10))/\sqrt{5},$$

where $R(n)$ denotes the range for one, n -round shot group. Substituting the CV 's for one, 3-round shot group and one, 10-round shot group (CV 's are listed in Table 3), and solving for k :

$$k = 5(0.525/0.259)^2 = 20.5;$$

thus, at least 21 shot groups are needed at a cost of expending 26% more rounds.

Example 6. Select the Average Range as the dispersion measure, and compute the 90% Confidence Interval of the ratio of the unbiased estimate of the population standard deviation ($\hat{\sigma}$) to σ (i.e., $\hat{\sigma}/\sigma$) for 3, 5-round shot groups.

Recall (11):

$$T_{0.05} \leq M/\sigma \leq T_{0.95}.$$

Dividing through by $\langle T \rangle$ and using (8):

$$T_{0.05}/\langle T \rangle \leq \hat{\sigma}/\sigma \leq T_{0.95}/\langle T \rangle. \quad (17)$$

Substituting the values in the last two columns of the Table 4 for 3, 5-round shot groups gives:

$$0.664 \leq \hat{\sigma}/\sigma \leq 1.369 \quad (90\% \text{ CI}).$$

Example 7. Repeat the computation in Example 6 for 5, 3-round shot groups. (Note the total number of rounds is the same as in the previous example.)

Following the procedure used in Example 6 and using the values in the last two columns of Table 4 for 5, 3-round shot groups:

$$0.634 \leq \hat{\sigma}/\sigma \leq 1.402 \quad (90\% \text{ CI}).$$

Comparing with Example 6, the overall 90% CI is ~9% larger if 5, 3-round groups are fired.

Example 8. Select the Average Range as the dispersion measure, and compute the 80% Confidence Interval of the ratio of $\hat{\sigma}/\sigma$ for 3, 5-round shot groups.

Referring to Appendix 1-2, the 10th and 90th percentiles of R/σ for 3, 5-round shot groups are 1.701 and 2.979, respectively. Analogous to (17):

$$T_{0.10}/\langle T \rangle \leq \hat{\sigma}/\sigma \leq T_{0.90}/\langle T \rangle.$$

Inserting the mean for 3, 5-round shot groups (Table 4) and the needed percentiles into the above inequality:

$$1.701/2.326 \leq \hat{\sigma}/\sigma \leq 2.979/2.326$$

$$0.731 \leq \hat{\sigma}/\sigma \leq 1.281 \quad (80\% \text{ CI}).$$

Example 9. If a total of N rounds is available to fire, what group size (n) gives the smallest Coefficient of Variation of the Average Range? (This example was inspired by Ref. 5.)

The number of groups is $k = N/n$, and the CV of AR for k, n -round groups is:

$$CV(AR(k, n)) = CV(R(n))/\sqrt{k}$$

$$CV(AR(k, n)) = \sqrt{\frac{n}{N}} CV(R(n)).$$

(See Example 5 for an explanation of the notation used above.) The value of n that minimizes $\sqrt{n}CV(R(n))$ gives the optimal group size, and computation of the optimal group size is illustrated in Table 5. The first two columns display the group size (n) and the CV (values are from Table 3). The third column shows the product of the value in the second column with the square root of the value in the first column, and the value of n for optimal group size corresponds to the minimum value in the third column. Finally, the fourth column shows an estimate of the Standard Error (SE) of the corresponding values in the third column. Figure 2 displays a plot of $\sqrt{n}CV$ versus n , and inspection shows a shallow minimum in the interval $6 < n < 11$, which indicates an optimal group size of about 8 for the univariate Range. From a practical standpoint, 10-round groups should serve just as well as 8-round groups to minimize CV .

Table 5. Table illustrating the computation of the optimal group size for the univariate Range.

n	CV	$\sqrt{n}CV$	SE
3	0.525	0.909	0.002
4	0.426	0.853	0.002
5	0.371	0.830	0.002
6	0.335	0.820	0.001
7	0.308	0.815	0.002
8	0.287	0.813	0.001
9	0.272	0.816	0.002
10	0.259	0.818	0.001
11	0.248	0.823	0.001
12	0.239	0.829	0.002
13	0.231	0.834	0.001
14	0.224	0.838	0.002
15	0.218	0.842	0.001

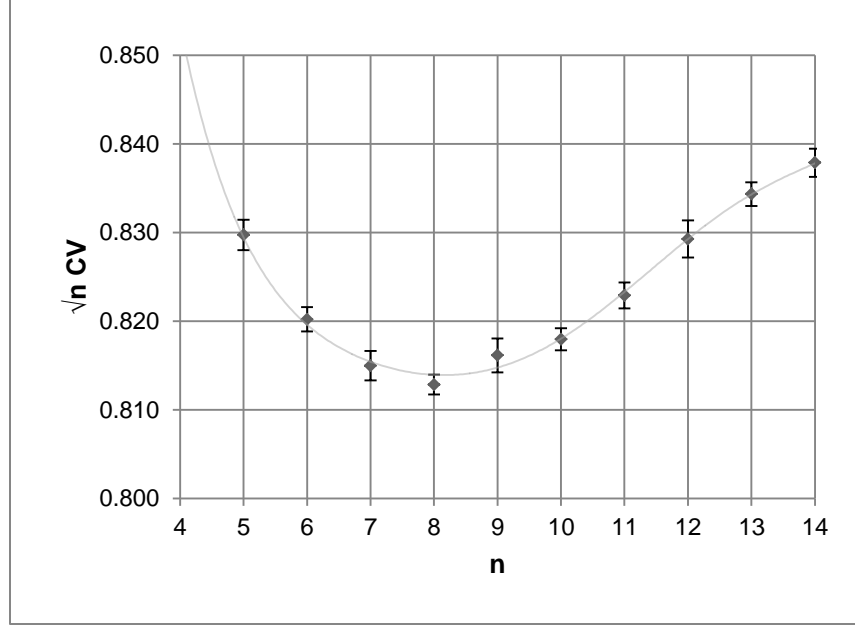


Figure 2. The optimal group size for the univariate Range is determined from the minimum observed in the plot of $\sqrt{n}CV$ versus n . The solid curve is a 6th order polynomial fit to guide the eye.

3.1.3. Mean Deviation, One Shot Group

The measure of dispersion known as the Mean Deviation is given by:

$$MD = \frac{1}{n} \sum_{i=1}^n |x_i - \bar{x}|, \quad (18)$$

which is the average of the distances of the x -components of the n impact points from the sample mean of x -components (\bar{x}). If the x -components are horizontal distances, then the MD is referred to as the Mean Horizontal Deviation (MHD), and if these are vertical distances, then it is referred to as the Mean Vertical Deviation (MVD).¹ Dividing (18) through by σ and rewriting:

$$MD/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}} \sum_{i=1}^n \sqrt{\left(\frac{\sqrt{n}(x_i - \bar{x})}{\sqrt{(n-1)}\sigma} \right)^2}. \quad (19)$$

Next, define:

$$u_i \equiv \frac{\sqrt{n}(x_i - \bar{x})}{\sqrt{(n-1)}\sigma}, \quad (20)$$

which is normally distributed with zero mean and unity variance. Making use of (20), (19) is rewritten as:

$$MD/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}} \sum_{i=1}^n |u_i|$$

$$MD/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}} \sum_{i=1}^n \chi_i(1), \quad (21)$$

where $\chi_i(1)$ is a chi-distributed random variable with one degree of freedom. For $n = 2$, we can write (21) as:

$$MD/\sigma = \frac{1}{\sqrt{2}}\chi(1) \quad (n = 2). \quad (22)$$

From (21), the mean of MD/σ is:

$$\begin{aligned} \langle MD/\sigma \rangle &= \frac{\sqrt{(n-1)}}{n\sqrt{\pi}} \sum_{i=1}^n \langle \chi_i(1) \rangle \\ &= \sqrt{\frac{(n-1)}{n}} \langle \chi(1) \rangle \\ &= \sqrt{\frac{(n-1)}{n}} \frac{\sqrt{2}}{\Gamma(\frac{1}{2})} \\ \langle MD/\sigma \rangle &= \sqrt{\frac{2(n-1)}{n\pi}}. \end{aligned} \quad (23)$$

In the limit of large n , $\langle MD/\sigma \rangle \rightarrow \sqrt{2/\pi}$, and for $n = 2$, (23) becomes:

$$\langle MD/\sigma \rangle = 1/\sqrt{\pi} \quad (n = 2). \quad (24)$$

Note (24) is in agreement with (5), since for this case $MD = \frac{1}{2}R$. Finally, we seek an analytical expression for the variance of MD/σ . Pearson and Hartley³ have provided such an expression, which is given by:

$$Var(MD/\sigma) = \frac{2(n-1)}{\pi n^2} \left[\frac{\pi}{2} + \sqrt{n(n-2)} - n + \sin^{-1} \frac{1}{n-1} \right]. \quad (25)$$

Recapping, analytical formulas as a function of sample size are provided for the first two moments of MD/σ . If $n = 2$, the probability distribution of MD/σ has a convenient analytical form (22); however, a simple extension of these results is unknown for $n > 2$, as evident from the fact that MD/σ is proportional to a sum of correlated chi random variables (21), which are correlated through their dependence on the sample mean (the u_i and u_j pairs (see (20)) are correlated). As such, Monte Carlo simulations were used to estimate the percentiles for $n > 2$.

In Table 6, results are presented for the means, standard deviations, and 5th and 95th percentiles of MD/σ for sample sizes of $n = 2$ -30. The means were computed using (23), and standard deviations were computed from the square root of the variance (25). For $n = 2$, the percentiles were determined from those of a chi-distributed random variable (one degree of freedom) scaled by $\frac{1}{\sqrt{2}}$ as shown in (22). For $n > 2$, the percentiles are Monte Carlo estimates. A listing of percentiles of MD/σ is found in Appendix 1-3, along with the *MCE*'s of the estimated percentiles. In earlier work, Pearson and Hartley³ published the upper and lower 10th percentiles of MD/σ ($n = 2$ -10) to three decimal places. Comparison of their results with the percentiles in Appendix 1-3 are in good agreement.

Table 6. Table of means, standard deviations, and 5th and 95th percentiles of MD/σ for sample sizes of $n = 2-30$.

n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
2	0.5642	0.4263	0.7555	0.0443	1.3859	0.0786	2.4565
3	0.6515	0.3419	0.5249	0.165	1.275	0.254	1.958
4	0.6910	0.2970	0.4298	0.256	1.221	0.370	1.767
5	0.7136	0.2663	0.3732	0.316	1.185	0.443	1.661
6	0.7284	0.2436	0.3344	0.360	1.157	0.494	1.588
7	0.7387	0.2258	0.3057	0.394	1.133	0.534	1.533
8	0.7464	0.2115	0.2834	0.423	1.115	0.566	1.494
9	0.7523	0.1996	0.2653	0.446	1.100	0.593	1.463
10	0.7569	0.1894	0.2503	0.464	1.087	0.613	1.436
11	0.7608	0.1807	0.2376	0.481	1.074	0.632	1.411
12	0.7639	0.1731	0.2266	0.495	1.063	0.647	1.391
13	0.7666	0.1664	0.2171	0.507	1.054	0.661	1.376
14	0.7689	0.1604	0.2086	0.519	1.045	0.675	1.360
15	0.7708	0.1550	0.2011	0.529	1.036	0.686	1.344
16	0.7725	0.1501	0.1943	0.537	1.030	0.696	1.333
17	0.7741	0.1457	0.1882	0.545	1.024	0.705	1.322
18	0.7754	0.1416	0.1826	0.552	1.018	0.712	1.312
19	0.7766	0.1378	0.1775	0.560	1.012	0.721	1.303
20	0.7777	0.1344	0.1728	0.565	1.007	0.727	1.295
21	0.7787	0.1312	0.1684	0.571	1.003	0.734	1.288
22	0.7795	0.1282	0.1644	0.577	0.998	0.741	1.280
23	0.7803	0.1254	0.1606	0.582	0.993	0.746	1.272
24	0.7811	0.1227	0.1571	0.586	0.989	0.750	1.267
25	0.7818	0.1203	0.1538	0.591	0.987	0.756	1.262
26	0.7824	0.1179	0.1507	0.596	0.983	0.761	1.256
27	0.7830	0.1157	0.1478	0.599	0.978	0.765	1.250
28	0.7835	0.1137	0.1451	0.603	0.977	0.769	1.247
29	0.7840	0.1117	0.1425	0.607	0.975	0.774	1.243
30	0.7845	0.1098	0.1400	0.610	0.970	0.778	1.237

Notes Table 6: For more details, refer to the notes at the bottom of Table 3.

3.1.4. Average Mean Deviation, Multiple Shot Groups

For multiple, n -round shot groups, the Mean Deviation is averaged over all groups to provide a measure of dispersion:

$$AMD/\sigma = \frac{1}{k} \sum_{i=1}^k MD_i/\sigma, \quad (26)$$

where AMD is the Average Mean Deviation, MD_i is the mean deviation of the i^{th} n -round shot group, and k is the total number of groups. Computing the mean of (26) gives:

$$\langle AMD/\sigma \rangle = \langle MD/\sigma \rangle; \quad (27)$$

i.e., the mean of an average over groups is the same as the mean of one shot group. The standard deviation of AMD/σ is:

$$Std(AMD/\sigma) = \frac{1}{\sqrt{k}} Std(MD/\sigma). \quad (28)$$

Analogous to (16), the CV is:

$$CV(AMD) = \frac{1}{\sqrt{k}} CV(MD). \quad (29)$$

In Table 7, results are presented for the means, standard deviations, and 5th and 95th percentiles of AMD/σ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$ for each of the listed n). The means were computed using (23), and the standard deviations were computed from the square root of the variance (25) and dividing by \sqrt{k} . For $n = 2$ with $k > 1$ or if $n > 2$, the percentiles are Monte Carlo estimates. A listing of the percentiles of AMD/σ for the various n and k is found in Appendix 1-4, along with the MCE 's of the estimated percentiles. As k increases, AMD/σ is asymptotically normal in accordance with the central limit theorem. As mentioned in Section 3.1.2, our Monte Carlo results on percentiles may be used to provide the error in approximating the probability distribution of AMD/σ by a normal distribution. We will not belabor this point any further in Section 3.2.2 on *AES* or Section 3.2.4 on *AMR*, but note that this approximation improves for these bivariate distributions.

Table 7. Table of means, standard deviations, and 5th and 95th percentiles of AMD/σ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$).

k	n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
1	2	0.5642	0.4263	0.7555	0.0443	1.3859	0.0786	2.4565
2			0.3014	0.5342	0.142	1.120	0.253	1.985
3			0.2461	0.4362	0.205	1.007	0.363	1.784
4			0.2131	0.3778	0.246	0.943	0.435	1.672
5			0.1906	0.3379	0.277	0.902	0.491	1.599
10			0.1348	0.2389	0.355	0.798	0.630	1.414
1	3	0.6515	0.3419	0.5249	0.165	1.275	0.254	1.958
2			0.2418	0.3711	0.288	1.079	0.443	1.656
3			0.1974	0.3030	0.349	1.000	0.535	1.534
4			0.1710	0.2624	0.387	0.948	0.594	1.455
5			0.1529	0.2347	0.413	0.915	0.634	1.405
10			0.1081	0.1660	0.480	0.835	0.737	1.282
1	4	0.6910	0.2970	0.4298	0.256	1.221	0.370	1.767
2			0.2100	0.3039	0.369	1.060	0.535	1.535
3			0.1715	0.2482	0.426	0.989	0.616	1.431
4			0.1485	0.2149	0.459	0.947	0.664	1.370
5			0.1328	0.1922	0.482	0.919	0.698	1.329
10			0.0939	0.1359	0.541	0.850	0.783	1.230
1	5	0.7136	0.2663	0.3732	0.316	1.185	0.443	1.661
2			0.1883	0.2639	0.422	1.042	0.591	1.459
3			0.1538	0.2155	0.473	0.976	0.663	1.367
4			0.1332	0.1866	0.504	0.940	0.706	1.318
5			0.1191	0.1669	0.525	0.916	0.736	1.283
10			0.0842	0.1180	0.579	0.855	0.812	1.199
1	10	0.7569	0.1894	0.2503	0.464	1.087	0.613	1.436
2			0.1340	0.1770	0.546	0.985	0.721	1.302
3			0.1094	0.1445	0.583	0.943	0.770	1.246
4			0.0947	0.1251	0.606	0.917	0.801	1.211
5			0.0847	0.1119	0.621	0.900	0.821	1.189
10			0.0599	0.0791	0.660	0.857	0.872	1.133
1	25	0.7818	0.1203	0.1538	0.591	0.987	0.756	1.262
2			0.0850	0.1088	0.645	0.924	0.826	1.182
3			0.0694	0.0888	0.670	0.898	0.857	1.149
4			0.0601	0.0769	0.685	0.882	0.876	1.128
5			0.0538	0.0688	0.694	0.872	0.888	1.115
10			0.0380	0.0486	0.720	0.845	0.921	1.081

Notes Table 7: For more details, refer to the notes at the bottom of Table 4.

Example 10. Determine the power curve of the Average Mean Deviation for 3, 5-round shot groups with $\alpha = 0.05$.

The power curve is generated by plotting the probability of rejection ($P_{rej}(T > (\frac{\sigma^*}{\sigma}) T_{1-\alpha})$) versus the ratio of σ/σ^* , where σ^* is the value of the population standard deviation that we do not wish to exceed for accepted product (see Appendix 2-1 for additional details).

For notational convenience, the probability distribution of AMD for 3, 5-round groups is denoted as $AMD(3,5)$, and here T (see (10)) is its probability distribution in units of σ (i.e., $T = AMD(3,5)/\sigma$). In Table 8, a tabular method is used to construct the power curve ($\alpha = 0.05$) of $AMD(3,5)$. The percentiles of $AMD(3,5)/\sigma$ are from Appendix 1-4, and the ratios of σ/σ^* (third column) are obtained by computing the ratio of the 95th percentile ($T_{1-\alpha}$) with each listed T_p . The fourth column displays the probability of rejection for a particular value of σ/σ^* and is computed by subtracting the values in the first column from one. Finally, the power curve is obtained by plotting the 4th versus 3rd column (see Figure 3).

Table 8. The tabular method used to construct the power curve ($\alpha = 0.05$) of AMD for 3, 5-round groups. Plotting the 4th versus the 3rd column generates the power curve.

$T = AMD(3, 5)/\sigma$			
p	T_p	σ/σ^*	P_{rej}
0.999	1.247	0.782	0.001
0.995	1.148	0.850	0.005
0.99	1.101	0.887	0.01
0.975	1.032	0.946	0.025
0.95	0.976	1.000	0.05
0.9	0.914	1.068	0.1
0.8	0.841	1.161	0.2
0.75	0.813	1.201	0.25
0.5	0.707	1.381	0.5
0.25	0.606	1.611	0.75
0.2	0.582	1.678	0.8
0.1	0.521	1.874	0.9
0.05	0.473	2.063	0.95
0.025	0.433	2.253	0.975
0.01	0.389	2.506	0.99
0.005	0.361	2.704	0.995

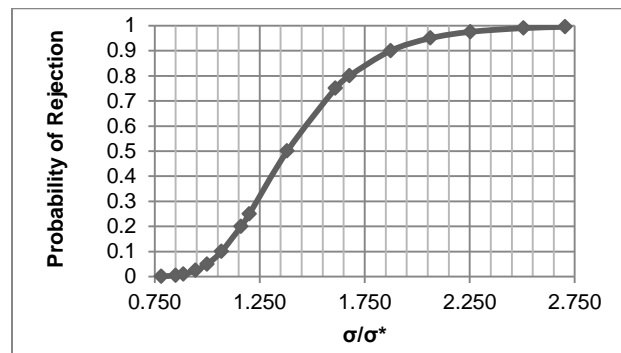


Figure 3. The power curve of the Average Mean Deviation for 3, 5-round shot groups with $\alpha = 0.05$.

Example 11. Using the results in Table 8, determine the probability of rejecting 3, 5-round groups when $\sigma = 2.5 \sigma^*$.

Inspection of Table 8 shows the probability of rejection is approximately 0.99.

Example 12. Suppose we shoot 10, 5-round groups and use the Average Mean Deviation as the dispersion measure. What is σ/σ^* for a rejection rate (power = $1 - \beta$) of 0.80 with $\alpha = 0.05$?

Following the method in Example 10, we seek the value of σ/σ^* for which:

$$\begin{aligned} \frac{\sigma}{\sigma^*} &= \frac{T_{1-\alpha}}{T_\beta} \\ &= \frac{T_{0.95}}{T_{0.20}}, \end{aligned}$$

where $T = AMD/\sigma$. Using the percentile values of AMD/σ for 10, 5-round shot groups (Appendix 1-4):

$$\frac{\sigma}{\sigma^*} = \frac{0.855}{0.642} = 1.332,$$

which demonstrates if $\sigma = 1.332 \sigma^*$, then the probability of rejection is 80%.

3.1.5. Standard Deviation, One Shot Group

The sample variance of the random variable x is given by:

$$SD^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2, \quad (30)$$

which is the average of the sum of squares of the deviations of the x -components of the impact points from the sample mean of x -components (\bar{x}). Taking the square root of both sides of (30) gives the univariate measure of dispersion known as the Standard Deviation:

$$SD = \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}. \quad (31)$$

If the x -components are horizontal distances, then SD is referred to as the Horizontal Standard Deviation (HSD), and if these are vertical distances, then it is referred to as the Vertical Standard Deviation (VSD). Dividing (31) by σ and rewriting:

$$\begin{aligned} SD/\sigma &= \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^n (x_i - \bar{x})^2 / \sigma^2} \\ SD/\sigma &= \frac{\chi(n-1)}{\sqrt{n}}, \end{aligned} \quad (32)$$

where $\chi(n-1)$ is a chi random variable with $(n-1)$ degrees of freedom. The mean of SD/σ is given by:

$$\langle SD/\sigma \rangle = \langle \chi(n-1) \rangle / \sqrt{n}$$

$$\langle SD/\sigma \rangle = \sqrt{\frac{2}{n}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})}, \quad (33)$$

and its variance by:

$$Var(SD/\sigma) = \frac{1}{n}[\langle \chi^2(n-1) \rangle - \langle \chi(n-1) \rangle^2]$$

$$Var(SD/\sigma) = \frac{1}{n} \left[(n-1) - 2 \left(\frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-1}{2})} \right)^2 \right]. \quad (34)$$

The above results demonstrate the advantage of using SD as a dispersion measure. The probability distribution of $\sqrt{n}SD/\sigma$ is a chi distribution whose percentiles are easily computed by taking the square root of the percentiles from the corresponding chi-square distribution. In addition, formulas for the first two moments are provided, which are easy to compute using the built-in functions in Microsoft Excel 2010.

In Table 9, results are presented for the means, standard deviations, and 5th and 95th percentiles of SD/σ for sample sizes of $n = 2-30$. The means and standard deviations were computed using (33) and (34). The percentiles were computed from those of a chi-distributed random variable $((n-1)$ DoF) scaled by $1/\sqrt{n}$ as shown in (32). A listing of additional percentiles of SD/σ is found in Appendix 1-5.

Table 9. Table of means, standard deviations, and 5th and 95th percentiles of SD/σ for sample sizes $n = 2-30$.

n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
2	0.5642	0.4263	0.7555	0.0443	1.3859	0.0786	2.4565
3	0.7236	0.3782	0.5227	0.1849	1.4132	0.2556	1.9530
4	0.7979	0.3367	0.4220	0.2966	1.3977	0.3717	1.7518
5	0.8407	0.3052	0.3630	0.3770	1.3775	0.4484	1.6384
6	0.8686	0.2808	0.3232	0.4369	1.3583	0.5030	1.5638
7	0.8882	0.2612	0.2941	0.4833	1.3412	0.5442	1.5100
8	0.9027	0.2452	0.2716	0.5205	1.3260	0.5766	1.4690
9	0.9139	0.2318	0.2536	0.5510	1.3126	0.6030	1.4363
10	0.9227	0.2203	0.2388	0.5766	1.3007	0.6249	1.4096
11	0.9300	0.2104	0.2262	0.5985	1.2901	0.6436	1.3872
12	0.9359	0.2017	0.2155	0.6174	1.2805	0.6597	1.3681
13	0.9410	0.1940	0.2061	0.6340	1.2718	0.6738	1.3515
14	0.9453	0.1871	0.1979	0.6487	1.2638	0.6863	1.3370
15	0.9490	0.1809	0.1906	0.6618	1.2566	0.6974	1.3241
16	0.9523	0.1753	0.1840	0.6737	1.2499	0.7074	1.3126
17	0.9551	0.1701	0.1781	0.6843	1.2437	0.7165	1.3022
18	0.9576	0.1654	0.1727	0.6941	1.2380	0.7248	1.2927
19	0.9599	0.1611	0.1678	0.7030	1.2327	0.7324	1.2841
20	0.9619	0.1570	0.1633	0.7112	1.2277	0.7394	1.2762
21	0.9638	0.1533	0.1591	0.7188	1.2230	0.7458	1.2690
22	0.9655	0.1498	0.1552	0.7259	1.2186	0.7518	1.2622
23	0.9670	0.1466	0.1516	0.7324	1.2145	0.7574	1.2560
24	0.9684	0.1435	0.1482	0.7385	1.2106	0.7627	1.2501
25	0.9696	0.1407	0.1451	0.7443	1.2069	0.7676	1.2447
26	0.9708	0.1380	0.1421	0.7497	1.2034	0.7722	1.2396
27	0.9719	0.1354	0.1393	0.7547	1.2001	0.7765	1.2348
28	0.9729	0.1330	0.1367	0.7595	1.1969	0.7806	1.2302
29	0.9739	0.1307	0.1342	0.7640	1.1939	0.7845	1.2259
30	0.9748	0.1285	0.1319	0.7683	1.1910	0.7882	1.2219

Notes Table 9: For more details, refer to the notes at the bottom of Table 3.

3.1.6. Standard Deviation, Multiple Shot Groups

Previously, we presented dispersion measures for multiple shot groups based on averaging the dispersion over groups. Similarly, such a dispersion measure may be developed for SD for multiple groups. However, we will depart from this usual procedure, and our dispersion measure for multiple groups will be the Root Mean Square (RMS) of the SD 's over groups. This departure is advantageous as explained below.

First, we present an analogous treatment to that given in Sections 3.1.2 and 3.1.4. Consider the dispersion measure given by the standard deviation averaged over all groups:

$$ASD/\sigma = \frac{1}{k} \sum_{i=1}^k SD_i / \sigma, \quad (35)$$

where ASD denotes the Average Standard Deviation, SD_i is the standard deviation of the i^{th} n -round shot group, and k is the total number of groups. Using (32), (35) is expressed as:

$$ASD/\sigma = \frac{1}{k\sqrt{n}} \sum_{i=1}^k \chi_i (n-1), \quad (36)$$

which shows that $\sqrt{n} ASD/\sigma$ is a group average over chi-distributed, independent random variables each with $(n-1)$ degrees of freedom. For $k > 1$, the probability distribution of ASD/σ is determined from the sum over chi-distributed variables and is not in the best form for easy computation of percentiles. However, we do acknowledge that the probability distribution of ASD/σ is asymptotically normal with increasing k , and its convergence to a normal distribution is expected to be fast. However, we depart from use of this dispersion measure in favor of $RMS-SD$, which is discussed below.

Next, we consider an alternative procedure for constructing a dispersion measure for multiple groups of SD . Instead of averaging over groups, a measure of dispersion for multiple, n -round, shot groups is constructed from the RMS of the SD 's over the k groups:

$$RMS-SD = \sqrt{\frac{1}{k} \sum_{i=1}^k SD_i^2}, \quad (37)$$

Dividing (37) by σ and making use of (32):

$$RMS-SD/\sigma = \frac{1}{\sqrt{kn}} \sqrt{\sum_{i=1}^k \chi_i^2 (n-1)}. \quad (38)$$

Note the sum of k , chi-square, independent variables each with $(n-1)$ degrees of freedom is a chi-square variable with $k(n-1)$ degrees of freedom; therefore (38) is rewritten:

$$RMS-SD/\sigma = \frac{1}{\sqrt{kn}} \sqrt{\chi^2(k(n-1))}$$

$$RMS-SD/\sigma = \frac{1}{\sqrt{kn}} \chi(k(n-1)). \quad (39)$$

Percentiles of $\sqrt{kn} \times RMS-SD/\sigma$ are determined from a well-known probability distribution (chi-distributed random variable with $k(n-1)$ degrees of freedom), which is why $RMS-SD$ is preferred to ASD (see (36)) as a dispersion measure. The percentile tables for $RMS-SD/\sigma$ for various values of k and n

were easily computed using the built-in functions in Microsoft Excel. Continuing, the mean of $RMS-SD/\sigma$ is given by:

$$\begin{aligned}\langle RMS-SD/\sigma \rangle &= \frac{1}{\sqrt{kn}} \langle \chi(k(n-1)) \rangle \\ \langle RMS-SD/\sigma \rangle &= \sqrt{\frac{2}{kn}} \frac{\Gamma(\frac{k(n-1)+1}{2})}{\Gamma(\frac{k(n-1)}{2})},\end{aligned}\tag{40}$$

and its variance is given by:

$$\begin{aligned}Var(RMS-SD/\sigma) &= \frac{1}{kn} Var(\chi(k(n-1))) \\ Var(RMS-SD/\sigma) &= \frac{1}{kn} \left\{ k(n-1) - 2 \left(\frac{\Gamma(\frac{k(n-1)+1}{2})}{\Gamma(\frac{k(n-1)}{2})} \right)^2 \right\}.\end{aligned}\tag{41}$$

The CV of $RMS-SD$ is computed from:

$$CV(RMS-SD) = \frac{\sqrt{k(n-1) - \langle \chi(k(n-1)) \rangle^2}}{\langle \chi(k(n-1)) \rangle},\tag{42}$$

which can be compared to the Coefficient of Variation of ASD :

$$CV(ASD) = \frac{1}{\sqrt{k}} \frac{\sqrt{(n-1) - \langle \chi((n-1)) \rangle^2}}{\langle \chi((n-1)) \rangle}\tag{43}$$

Comparison of the CV's given by (42) and (43) for the same k and n would show the CV of $RMS-SD$ is less than that of ASD , which demonstrates the CV of $RMS-SD$ is a more efficient estimator of σ . Albeit, the difference between CV's is small for the various (k, n) pairs.

Next in Table 10, results are presented for the means, standard deviations, and 5th and 95th percentiles of $RMS-SD/\sigma$ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$ for each of the listed n). The means and standard deviations were computed from (40) and the square root of (41), and the percentiles were computed from a chi distribution ($k(n-1)$ degrees of freedom) scaled by $1/\sqrt{kn}$ per (39). A listing of additional percentiles of $RMS-SD/\sigma$ for the various k and n is found in Appendix 1-6.

Table 10. Table of means, standard deviations, and 5th and 95th percentiles of $RMS-SD/\sigma$ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$).

k	n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
1	2	0.5642	0.4263	0.7555	0.0443	1.3859	0.0786	2.4565
2		0.6267	0.3276	0.5227	0.1601	1.2239	0.2556	1.9530
3		0.6515	0.2749	0.4220	0.2422	1.1413	0.3717	1.7518
4		0.6647	0.2413	0.3630	0.2981	1.0890	0.4484	1.6384
5		0.6728	0.2175	0.3232	0.3384	1.0522	0.5030	1.5638
10		0.6897	0.1560	0.2262	0.4439	0.9567	0.6436	1.3872
1	3	0.7236	0.3782	0.5227	0.1849	1.4132	0.2556	1.9530
2		0.7675	0.2786	0.3630	0.3442	1.2575	0.4484	1.6384
3		0.7833	0.2304	0.2941	0.4263	1.1828	0.5442	1.5100
4		0.7914	0.2007	0.2536	0.4772	1.1368	0.6030	1.4363
5		0.7964	0.1802	0.2262	0.5125	1.1047	0.6436	1.3872
10		0.8064	0.1283	0.1591	0.6014	1.0232	0.7458	1.2690
1	4	0.7979	0.3367	0.4220	0.2966	1.3977	0.3717	1.7518
2		0.8308	0.2444	0.2941	0.4521	1.2546	0.5442	1.5100
3		0.8423	0.2011	0.2388	0.5264	1.1874	0.6249	1.4096
4		0.8482	0.1749	0.2061	0.5715	1.1464	0.6738	1.3515
5		0.8517	0.1567	0.1840	0.6025	1.1179	0.7074	1.3126
10		0.8588	0.1113	0.1296	0.6799	1.0461	0.7917	1.2180
1	5	0.8407	0.3052	0.3630	0.3770	1.3775	0.4484	1.6384
2		0.8670	0.2199	0.2536	0.5227	1.2453	0.6030	1.4363
3		0.8760	0.1806	0.2061	0.5903	1.1840	0.6738	1.3515
4		0.8806	0.1568	0.1781	0.6309	1.1467	0.7165	1.3022
5		0.8833	0.1405	0.1591	0.6588	1.1209	0.7458	1.2690
10		0.8889	0.0997	0.1121	0.7281	1.0560	0.8192	1.1881
1	10	0.9227	0.2203	0.2388	0.5766	1.3007	0.6249	1.4096
2		0.9356	0.1570	0.1678	0.6852	1.2014	0.7324	1.2841
3		0.9399	0.1285	0.1367	0.7337	1.1563	0.7806	1.2302
4		0.9421	0.1114	0.1183	0.7627	1.1291	0.8096	1.1985
5		0.9434	0.0997	0.1057	0.7825	1.1105	0.8294	1.1770
10		0.9461	0.0706	0.0746	0.8314	1.0637	0.8788	1.1244
1	25	0.9696	0.1407	0.1451	0.7443	1.2069	0.7676	1.2447
2		0.9747	0.0997	0.1023	0.8136	1.1417	0.8347	1.1713
3		0.9764	0.0815	0.0835	0.8443	1.1124	0.8647	1.1393
4		0.9772	0.0706	0.0723	0.8626	1.0949	0.8826	1.1203
5		0.9778	0.0632	0.0646	0.8750	1.0828	0.8949	1.1075
10		0.9788	0.0447	0.0457	0.9058	1.0529	0.9255	1.0757

Notes Table 10: For more details, refer to the notes at the bottom of Table 4.

3.2. Bivariate Dispersion Measures

3.2.1. Extreme Spread, One Shot Group

The bivariate measure of dispersion known as the Extreme Spread is given by:

$$ES = \max_{i,j} \left\{ \sqrt{(\vec{x}_i - \vec{x}_j)^2} \right\}, \quad (44)$$

where the vector $\vec{x}_i = (x_i, y_i)$ is the i^{th} impact-point position. ES is the maximum of the distances between all possible pairs of impact points. Dividing through by σ :

$$ES/\sigma = \max_{i,j} \left\{ \sqrt{((\vec{x}_i - \vec{x}_j)/\sigma)^2} \right\}, \quad (45)$$

where σ is the population standard deviation of x and y of the impact-point positions. For $n = 2$, (45) becomes:

$$ES/\sigma = |\vec{x}_2 - \vec{x}_1|/\sigma,$$

and is rewritten as:

$$ES/\sigma = \sqrt{2} \sqrt{\left(\frac{x_2 - x_1}{\sqrt{2}\sigma}\right)^2 + \left(\frac{y_2 - y_1}{\sqrt{2}\sigma}\right)^2} \quad (n = 2). \quad (46)$$

Since $\frac{x_2 - x_1}{\sqrt{2}\sigma}$ and $\frac{y_2 - y_1}{\sqrt{2}\sigma}$ are normally distributed with zero mean and unity variance, (46) is expressed as:

$$ES/\sigma = \sqrt{2}\chi(2) \quad (n = 2), \quad (47)$$

where $\chi(2)$ is a chi-distributed random variable with two degrees of freedom. Continuing with $n = 2$, the mean of ES/σ is given by:

$$\begin{aligned} \langle ES/\sigma \rangle &= \sqrt{2} \langle \chi(2) \rangle \\ &= 2\Gamma\left(\frac{3}{2}\right) \\ \langle ES/\sigma \rangle &= \sqrt{\pi} \quad (n = 2), \end{aligned} \quad (48)$$

and its variance is computed from:

$$\begin{aligned} Var(ES/\sigma) &= \langle (ES/\sigma)^2 \rangle - \langle ES/\sigma \rangle^2 \\ &= 2[\langle \chi^2(2) \rangle - \pi/2] \\ Var(ES/\sigma) &= (4 - \pi) \quad (n = 2). \end{aligned} \quad (49)$$

Taking the square root of (49) gives the standard deviation:

$$Std(ES/\sigma) = \sqrt{(4 - \pi)} \quad (n = 2). \quad (50)$$

For $n = 2$, simple analytical results are provided for the probability distribution and the first two moments of ES/σ . If $n > 2$, an extension of these analytic results is unavailable, because the exact probability distributions of ES are unknown. In this case, Monte Carlo methods are an obvious approach to compute the percentiles and moments.²

In Table 11, results are presented for the means, standard deviations, and 5th and 95th percentiles of ES/σ for sample sizes of $n = 2$ -30. For $n = 2$, the means and standard deviations displayed in the table are from (48) and (50), and the percentiles were determined from those of a chi-distributed random variable (two degrees of freedom) scaled by $\sqrt{2}$ as shown in (47). For $n > 2$, all tabulated results are from Monte Carlo estimates. A listing of additional percentiles of ES/σ is found in Appendix 1-7, along with the *MCE*'s of the estimated percentiles. In previous work, Taylor and Grubbs² examined the probability distribution of

the Extreme Spread for various sample sizes ($n = 3-15, 20, 25, 28, 30, 31, 34$) using Monte Carlo methods and published percentiles of ES along with its first four moments. Our results presented in Table 11 and Appendix 1-7 are in excellent agreement with theirs—our published values of percentiles, means, and standard deviations agree within 1%.

Table 11. Table of means, standard deviations, 5th and 95th percentiles of ES/σ for sample sizes of $n = 2-30$. Numbers reported in parentheses are the MCE 's of the value of the mean or standard deviation preceding it.

n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
2	1.7725	0.9265	0.5227	0.4530	3.4616	0.2556	1.9530
3	2.410 (0.008)	0.893 (0.006)	0.370	1.067	3.986	0.443	1.654
4	2.795 (0.004)	0.855 (0.004)	0.306	1.497	4.291	0.535	1.535
5	3.062 (0.006)	0.828 (0.004)	0.270	1.804	4.515	0.589	1.474
6	3.275 (0.005)	0.803 (0.002)	0.245	2.048	4.679	0.625	1.429
7	3.443 (0.005)	0.782 (0.003)	0.227	2.246	4.811	0.652	1.397
8	3.584 (0.004)	0.766 (0.004)	0.214	2.411	4.924	0.673	1.374
9	3.705 (0.004)	0.752 (0.002)	0.203	2.556	5.015	0.690	1.354
10	3.815 (0.005)	0.741 (0.004)	0.194	2.686	5.106	0.704	1.339
11	3.904 (0.007)	0.729 (0.003)	0.187	2.791	5.176	0.715	1.326
12	3.989 (0.005)	0.719 (0.004)	0.180	2.893	5.250	0.725	1.316
13	4.064 (0.006)	0.711 (0.004)	0.175	2.983	5.312	0.734	1.307
14	4.131 (0.005)	0.704 (0.004)	0.170	3.062	5.360	0.741	1.297
15	4.197 (0.005)	0.698 (0.003)	0.166	3.133	5.424	0.746	1.292
16	4.257 (0.003)	0.692 (0.003)	0.163	3.204	5.474	0.753	1.286
17	4.310 (0.005)	0.685 (0.005)	0.159	3.266	5.505	0.758	1.277
18	4.360 (0.006)	0.679 (0.004)	0.156	3.327	5.554	0.763	1.274
19	4.407 (0.005)	0.675 (0.003)	0.153	3.383	5.592	0.768	1.269
20	4.451 (0.004)	0.670 (0.003)	0.150	3.438	5.623	0.772	1.263
21	4.495 (0.007)	0.665 (0.003)	0.148	3.488	5.658	0.776	1.259
22	4.533 (0.005)	0.660 (0.004)	0.146	3.533	5.696	0.780	1.257
23	4.571 (0.005)	0.656 (0.004)	0.144	3.578	5.720	0.783	1.251
24	4.606 (0.005)	0.654 (0.004)	0.142	3.615	5.757	0.785	1.250
25	4.640 (0.004)	0.650 (0.003)	0.140	3.657	5.784	0.788	1.247
26	4.672 (0.004)	0.645 (0.002)	0.138	3.697	5.809	0.791	1.243
27	4.704 (0.007)	0.642 (0.004)	0.137	3.732	5.831	0.793	1.240
28	4.733 (0.006)	0.639 (0.004)	0.135	3.767	5.859	0.796	1.238
29	4.763 (0.005)	0.637 (0.005)	0.134	3.802	5.884	0.798	1.235
30	4.790 (0.004)	0.634 (0.003)	0.132	3.837	5.908	0.801	1.233

Notes Table 11: For more details, refer to the notes at the bottom of Table 3.

Example 13. Compute an unbiased estimate of σ using the Extreme Spread of the 5-round shot group in Table 1.

First compute the ES by substituting the x and y values of the impact points (Table 1) into (44), which gives $ES = 1.79''$. An unbiased estimate is computed from:

$$\hat{\sigma} = ES / \langle ES/\sigma \rangle.$$

Substituting the needed mean (Table 11) and computed ES :

$$\hat{\sigma} = \frac{1.79''}{3.062} = 0.58''.$$

Comparison with Example 1 shows a 16% reduction in $\hat{\sigma}$ when the bivariate measure is used.

Example 14. Using the Extreme Spread computed in Example 13, find the 90% Confidence Interval of σ from this 5-round group.

Recall (12):

$$M/T_{0.95} \leq \sigma \leq M/T_{0.05}.$$

Substituting the *ES* of the group and the needed values for a 5-round group (Table 11):

$$1.79"/4.515 \leq \sigma \leq 1.79"/1.804$$

$$0.40" \leq \sigma \leq 0.99" \quad (90\% \text{ CI}).$$

Compare this result with Example 3, where the 90% CI was computed using the univariate Range. Note the 95% upper CL is reduced by ~36% when the bivariate measure of dispersion is used.

3.2.2. Extreme Spread, Multiple Shot Groups

For multiple, n -round shot groups, the Extreme Spread is averaged over groups to provide a measure of dispersion:

$$AES/\sigma = \frac{1}{k} \sum_{i=1}^k ES_i / \sigma, \quad (51)$$

where *AES* is the Average Extreme Spread, ES_i is the extreme spread of the i^{th} n -round shot group, and k is the total number of groups. The mean of (51) is:

$$\langle AES/\sigma \rangle = \langle ES/\sigma \rangle, \quad (52)$$

and its standard deviation is:

$$Std(AES/\sigma) = \frac{1}{\sqrt{k}} Std(ES/\sigma), \quad (53)$$

and its Coefficient of Variation is:

$$CV(AES) = \frac{1}{\sqrt{k}} CV(ES). \quad (54)$$

In Table 12, results are presented for the means, standard deviations, and 5th and 95th percentiles of AES/σ for n -round shot groups ($n = 2, 5, 10, 25$) fired for a total of k groups ($k = 1, 5, 10$ for each of the listed n). For $n = 2$, the listed mean equals $\sqrt{\pi}$ via (48), and the standard deviations are computed from (50) divided by \sqrt{k} . For $n = 2$ and $k > 1$, the percentiles are Monte Carlo estimates. For $n > 2$, all tabulated results are Monte Carlo estimates, and estimates of the means are from simulations with $k = 10$. A listing of the percentiles of AES/σ for the various n and k is found in Appendix 1-8, along with the *MCE*'s of the estimated percentiles.

Table 12. Table of means, standard deviations, and 5th and 95th percentiles of AES/σ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$). Numbers reported in parentheses are the MCE 's of the value of the mean or standard deviation preceding it.

k	n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
1	2	1.7725	0.9265	0.5227	0.4530	3.4616	0.2556	1.9530
2			0.6551	0.3696	0.788	2.934	0.445	1.656
3			0.5349	0.3018	0.952	2.711	0.537	1.529
4			0.4633	0.2614	1.056	2.579	0.596	1.455
5			0.4143	0.2338	1.126	2.487	0.635	1.403
10			0.2930	0.1653	1.307	2.269	0.738	1.280
1	3	2.409 (0.003)	0.893 (0.006)	0.371	1.067	3.986	0.443	1.655
2			0.632 (0.004)	0.262	1.428	3.502	0.593	1.454
3			0.515 (0.002)	0.214	1.602	3.291	0.665	1.367
4			0.446 (0.002)	0.185	1.706	3.173	0.708	1.317
5			0.400 (0.0019)	0.166	1.775	3.092	0.737	1.284
10			0.282 (0.0008)	0.117	1.957	2.884	0.812	1.197
1	4	2.794 (0.0013)	0.855 (0.004)	0.306	1.497	4.291	0.536	1.536
2			0.605 (0.004)	0.217	1.850	3.837	0.662	1.373
3			0.494 (0.002)	0.177	2.016	3.638	0.722	1.302
4			0.428 (0.003)	0.153	2.113	3.520	0.756	1.260
5			0.383 (0.0018)	0.137	2.183	3.443	0.781	1.232
10			0.271 (0.0016)	0.097	2.359	3.250	0.845	1.163
1	5	3.066 (0.002)	0.828 (0.004)	0.270	1.804	4.515	0.588	1.473
2			0.586 (0.003)	0.191	2.154	4.077	0.703	1.330
3			0.477 (0.002)	0.156	2.311	3.881	0.754	1.266
4			0.413 (0.0015)	0.135	2.409	3.765	0.786	1.228
5			0.370 (0.0014)	0.121	2.477	3.691	0.808	1.204
10			0.262 (0.0012)	0.086	2.644	3.505	0.862	1.143
1	10	3.811 (0.0014)	0.741 (0.004)	0.194	2.686	5.106	0.705	1.340
2			0.524 (0.004)	0.137	2.994	4.710	0.786	1.236
3			0.428 (0.002)	0.112	3.136	4.542	0.823	1.192
4			0.371 (0.0012)	0.097	3.224	4.442	0.846	1.166
5			0.331 (0.002)	0.087	3.285	4.373	0.862	1.148
10			0.234 (0.0011)	0.061	3.434	4.204	0.901	1.103
1	25	4.641 (0.0018)	0.650 (0.003)	0.140	3.657	5.784	0.788	1.246
2			0.458 (0.0012)	0.099	3.927	5.430	0.846	1.170
3			0.375 (0.003)	0.081	4.051	5.283	0.873	1.138
4			0.324 (0.0017)	0.070	4.130	5.192	0.890	1.119
5			0.291 (0.0016)	0.063	4.179	5.133	0.900	1.106
10			0.206 (0.0013)	0.0443	4.311	4.987	0.929	1.075

Notes Table 12: For more details, refer to the notes at the bottom of Table 4.

Example 15. If a total of N rounds is available to fire, what group size (n) gives the smallest Coefficient of Variation of the Average Extreme Spread? (This example was inspired by Ref. 5.)

Following Example 9, the number of groups is $k = N/n$, and the CV of AES for k, n -round groups is:

$$CV(AES(k, n)) = CV(ES(n))/\sqrt{k}$$

$$CV(AES(k, n)) = \sqrt{\frac{n}{N}} CV(ES(n)).$$

The value of n that minimizes $\sqrt{n}CV(ES(n))$ gives the optimal group size, and computation of the optimal group size is illustrated in Table 13. The value of n for optimal group size corresponds to the minimum value in the third column, and the fourth column shows an estimate of the Standard Error (SE) of the corresponding values in the third column. Figure 4 shows a plot of $\sqrt{n}CV$ versus n , and inspection of the figure shows a shallow minimum in the interval $5 < n < 8$ indicating an optimal group size for the Extreme Spread of about $n = 7$. From a practical standpoint, 5-round groups should serve just as well as 7-round groups to minimize CV .

Table 13. Table illustrating the computation of the optimal group size for the Extreme Spread.

n	CV	$\sqrt{n}CV$	SE
3	0.370	0.642	0.002
4	0.306	0.612	0.001
5	0.270	0.604	0.001
6	0.245	0.600	0.001
7	0.227	0.600	0.001
8	0.214	0.605	0.001
9	0.203	0.609	0.001
10	0.194	0.614	0.001
11	0.187	0.619	0.001
12	0.180	0.625	0.001
13	0.175	0.630	0.001
14	0.170	0.637	0.001
15	0.166	0.644	0.001

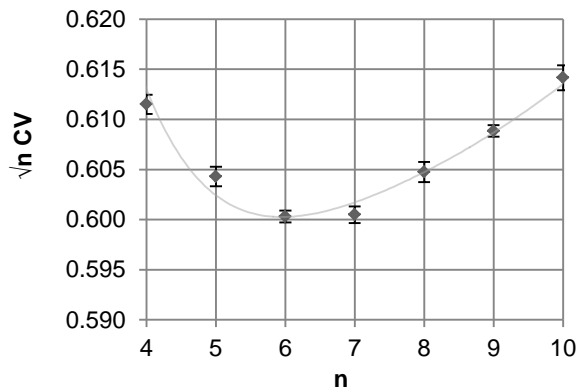


Figure 4. The optimal group size for the Extreme Spread is determined from the minimum observed in the plot of $\sqrt{n}CV$ versus n . The solid curve is a 6th order polynomial fit to guide the eye.

Example 16. Suppose 3, 5-round shot groups are subjected to the requirement that the Average Extreme Spread over the three groups shall be less than 1 MOA ($AES \leq 1 \text{ MOA}$). What is the upper limit on σ (σ_{upper}) at the 95% CL for acceptance? What is the Circular Error Probability (CEP) at the 95% CL?

Inspection of (12) shows:

$$\sigma_{upper} = 1 \text{ MOA} / T_{0.05} \quad (95\% \text{ CL}).$$

Here $T_{0.05}$ represents the 5th percentile of AES/σ . Substituting the value of this percentile for 3, 5-round groups (Table 12):

$$\sigma_{upper} = 1 \text{ MOA} / 2.311 = 0.433 \text{ MOA} \quad (95\% \text{ CL}). \quad (55)$$

The CEP is defined as:

$$CEP \equiv \sqrt{2 \ln 2} \sigma_{upper}, \quad (56)$$

which is the radius of the circle in which 50% of the shots will fall at the CL of σ_{upper} . Substituting (55) into (56):

$$CEP = 0.509 \text{ MOA} \quad (95\% \text{ CL}).$$

Example 17. Another test agency decides to shoot 3, 5-round groups and use the maximum Extreme Spread of the three groups as their dispersion measure:

$$ES_{max} = \text{Max}(ES_1, ES_2, ES_3).$$

What should the requirement on ES_{max} be to ensure σ_{upper} is the same as given in Example 16?

Our requirement is:

$$ES_{max} \leq Q,$$

where Q is the quantity that we are seeking to determine. Accordingly, we have:

$$\sigma_{upper} = Q / T'_{0.05} \quad (95\% \text{ CL}), \quad (57)$$

where $T'_{0.05}$ represents the 5th percentile of ES_{max}/σ . This percentile needs to be computed, and we note that it is equivalent to some $p \times 100$ percentile of ES/σ for one, 5-round group. Since there are 3 groups from which the maximum ES is selected:

$$p = (0.05)^{\frac{1}{3}} = 0.368.$$

So, we are seeking the value of $T_{0.368}$ for ES/σ for one, 5-round group, which we determine by linear interpolation between the 25th and 50th percentiles given in Appendix 1-7. Interpolation gives:

$$T'_{0.05} = 2.725.$$

To satisfy (57):

$$Q = T'_{0.05} \times \sigma_{upper} = 2.725 \times 0.433 \text{ MOA} = 1.18 \text{ MOA};$$

therefore, our requirement is:

$$ES_{max} \leq 1.18 \text{ MOA}.$$

Example 18. Compare the power curves of ES_{max} (defined in Example 17) and the Average Extreme Spread for 3, 5-round shot groups with $\alpha = 0.05$.

Following the procedure in Example 10, construct the power curves ($\alpha = 0.05$) of AES and ES_{max} for 3, 5-round groups. The power curves are plotted in Figure 5, which shows that σ/σ^* at a rejection rate of 0.80 is ~9% smaller for AES/σ . For clarity, Table 14 illustrates the tabular method used to construct the power curve of $ES_{max}(3,5)$. The percentiles of $ES_{max}(3,5)/\sigma$ were computed using the percentiles of ES/σ for one, 5-round group $ES(5)/\sigma$. For example, the value of 6.04 is the 99.9 percentile of $ES(5)/\sigma$ (Appendix 1-7), which is thus the value of the $(0.999)^{\frac{1}{3}} \times 100 = 99.7$ percentile of $ES_{max}(3,5)/\sigma$. The other percentiles were similarly computed with exception to the 20th and 95th percentiles, which were computed by linear interpolation using the corresponding adjacent percentiles computed for $ES_{max}(3,5)/\sigma$ in Table 14.

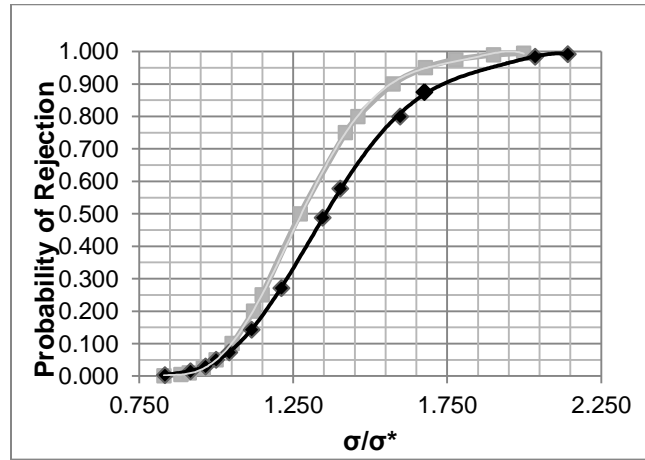


Figure 5. A comparison of the power curves ($\alpha = 0.05$) of AES (gray squares) and ES_{max} (black diamonds) for 3, 5-round groups.

Table 14. The tabular method used to construct the power curve ($\alpha = 0.05$) of ES_{max} for 3, 5-round groups. Plotting the 4th versus the 3rd column generates the power curve.

$T' = ES_{max}(3,5)/\sigma$			
p	T'_p	σ/σ^*	P_{rej}
0.997	6.04	0.83	0.003
0.985	5.49	0.92	0.015
0.970	5.22	0.97	0.030
0.950	5.04	1.00	0.050
0.927	4.83	1.04	0.073
0.857	4.515	1.12	0.143
0.729	4.156	1.21	0.271
0.512	3.743	1.35	0.488
0.422	3.589	1.40	0.578
0.200	3.15	1.60	0.800
0.125	3.005	1.68	0.875
0.016	2.474	2.04	0.984
0.008	2.352	2.14	0.992

3.2.3. Mean Radius, One Shot Group

The measure of dispersion known as the Mean Radius is given by:

$$MR = \frac{1}{n} \sum_{i=1}^n \sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}, \quad (58)$$

which is the average of the radial distances of the n impact points from the observed center of impact (\bar{x}, \bar{y}) . Dividing both sides of (58) by σ and rewriting:

$$MR/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}} \sum_{i=1}^n \sqrt{\left(\frac{\sqrt{n}(x_i - \bar{x})}{\sqrt{(n-1)}\sigma}\right)^2 + \left(\frac{\sqrt{n}(y_i - \bar{y})}{\sqrt{(n-1)}\sigma}\right)^2}. \quad (59)$$

Recall the definition given in (20):

$$u_i \equiv \frac{\sqrt{n}(x_i - \bar{x})}{\sqrt{(n-1)}\sigma}. \quad (60)$$

Similarly, we define:

$$v_i \equiv \frac{\sqrt{n}(y_i - \bar{y})}{\sqrt{(n-1)}\sigma}. \quad (61)$$

The u_i and v_i are normally distributed with zero mean and unity variance; therefore, substituting (60) and (61) into (59) allows us to write:

$$MR/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}} \sum_{i=1}^n \sqrt{\chi_i^2(2)},$$

where we have used the fact that the sum of two independent chi-square random variables with one degree of freedom is a chi-square variable with two degrees of freedom. Next, rewrite the above as:

$$MR/\sigma = \frac{\sqrt{(n-1)}}{n\sqrt{n}} \sum_{i=1}^n \chi_i(2). \quad (62)$$

For $n = 2$, we can write (62) as:

$$MR/\sigma = \frac{1}{\sqrt{2}} \chi(2) \quad (n = 2). \quad (63)$$

From (62), the mean of MR/σ is:

$$\begin{aligned} \langle MR/\sigma \rangle &= \frac{\sqrt{(n-1)}}{n\sqrt{n}} \sum_{i=1}^n \langle \chi_i(2) \rangle \\ &= \sqrt{\frac{n-1}{n}} \langle \chi(2) \rangle \\ &= \sqrt{\frac{2(n-1)}{n}} \Gamma\left(\frac{3}{2}\right) \\ \langle MR/\sigma \rangle &= \sqrt{\frac{(n-1)\pi}{2n}}. \end{aligned} \quad (64)$$

In the limit of large n , $\langle MR/\sigma \rangle \rightarrow \sqrt{\frac{\pi}{2}}$, and for $n = 2$, (64) becomes:

$$\langle MR/\sigma \rangle = \frac{\sqrt{\pi}}{2} \quad (n = 2), \quad (65)$$

which agrees with (48), since $MR = \frac{1}{2}ES$ for this special case. Next, we seek an analytical expression for the variance of MR/σ . Grubbs¹ has provided such an expression for the variance of MR/σ for $n > 2$:

$$Var(MR/\sigma) = \frac{2(n-1)}{n^2} \left\{ 1 - \frac{n\pi}{4} + \frac{(n-1)}{2} \psi \right\} \quad (n > 2), \quad (66)$$

where:

$$\psi = \left[1 - \left(\frac{1}{n-1} \right)^2 \right]^2 \sum_{j=0}^{\infty} \frac{\left(\frac{1}{n-1} \right)^j 2^{j+1} \left[\Gamma\left(\frac{j+3}{2}\right) \right]^2}{j!} \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \text{ is odd} \\ \frac{(j-1)!!}{j!!} & \text{if } j \text{ is even} \end{cases}. \quad (67)$$

For $n = 2$, the variance of MR/σ is computed directly from (63):

$$Var(MR/\sigma) = \frac{1}{2} Var(\chi(2))$$

$$Var(MR/\sigma) = \left(1 - \frac{\pi}{4} \right) \quad (n = 2). \quad (68)$$

Table 15. Table of means, standard deviations, and 5th and 95th percentiles of MR/σ for sample sizes of $n = 2$ -30.

n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
2	0.8862	0.4633	0.5227	0.2265	1.7308	0.2556	1.9530
3	1.0233	0.3738	0.3653	0.459	1.683	0.448	1.645
4	1.0854	0.3243	0.2988	0.586	1.651	0.539	1.521
5	1.1210	0.2906	0.2592	0.669	1.623	0.597	1.448
6	1.1441	0.2656	0.2322	0.729	1.602	0.637	1.400
7	1.1603	0.2461	0.2121	0.772	1.583	0.666	1.364
8	1.1724	0.2304	0.1965	0.809	1.565	0.690	1.335
9	1.1816	0.2174	0.1839	0.839	1.552	0.710	1.314
10	1.1890	0.2063	0.1735	0.862	1.541	0.725	1.296
11	1.1950	0.1968	0.1647	0.882	1.531	0.738	1.281
12	1.2000	0.1885	0.1571	0.900	1.519	0.750	1.266
13	1.2041	0.1811	0.1504	0.916	1.511	0.761	1.255
14	1.2077	0.1746	0.1445	0.930	1.505	0.770	1.246
15	1.2108	0.1687	0.1393	0.941	1.496	0.778	1.235
16	1.2135	0.1633	0.1346	0.954	1.489	0.786	1.227
17	1.2159	0.1585	0.1304	0.962	1.484	0.791	1.220
18	1.2180	0.1541	0.1265	0.972	1.477	0.798	1.213
19	1.2199	0.1500	0.1229	0.980	1.472	0.803	1.207
20	1.2216	0.1462	0.1197	0.986	1.468	0.807	1.202
21	1.2231	0.1427	0.1166	0.994	1.464	0.812	1.197
22	1.2245	0.1394	0.1138	1.002	1.461	0.818	1.193
23	1.2258	0.1364	0.1112	1.007	1.455	0.821	1.187
24	1.2269	0.1335	0.1088	1.011	1.451	0.824	1.183
25	1.2280	0.1308	0.1065	1.018	1.449	0.829	1.180
26	1.2290	0.1283	0.1044	1.023	1.444	0.833	1.175
27	1.2299	0.1259	0.1024	1.028	1.442	0.836	1.173
28	1.2307	0.1236	0.1004	1.032	1.438	0.838	1.168
29	1.2315	0.1215	0.0986	1.036	1.436	0.841	1.166
30	1.2322	0.1194	0.0969	1.039	1.432	0.843	1.162

Notes Table 15: For more details, refer to the notes at the bottom of Table 3.

From (68), the standard deviation is:

$$Std(MR/\sigma) = \sqrt{\left(1 - \frac{\pi}{4}\right)} \quad (n = 2). \quad (69)$$

In the above, analytical formulas as a function of sample size are provided for the first two moments of MR/σ . If $n = 2$, the probability distribution of MR/σ has a convenient analytical form (63); however, a simple extension of this analytical result is unavailable for $n > 2$, as evident from, the fact that MR/σ is proportional to a sum of correlated chi random variables¹ (the u_i and u_j pairs (see (60)) are correlated, and the v_i and v_j pairs (see (61)) are correlated). As such, Monte Carlo simulations were used to estimate the percentiles for $n > 2$.

In Table 15, results are presented for the means, standard deviations, and 5th and 95th percentiles of MR/σ for sample sizes of $n = 2-30$. Means were computed from (64), and standard deviations were computed from (69) for $n = 2$ and from the square root of the variance (see (66) and (67)) for $n > 2$. For $n = 2$, the percentiles were determined from those of a chi-distributed random variable (two degrees of freedom) scaled by $\frac{1}{\sqrt{2}}$ as shown in (63). However, for sample sizes $n > 2$, the percentiles are Monte Carlo estimates. A listing of percentiles of MR/σ is found in Appendix 1-9, along with the *MCE*'s of the estimated percentiles.

3.2.4. Average Mean Radius, Multiple Shot Groups

For multiple, n -round shot groups, the Mean Radius is averaged over all groups to provide a measure of dispersion:

$$AMR/\sigma = \frac{1}{k} \sum_{i=1}^k MR_i/\sigma \quad (70)$$

where AMR is the Average Mean Radius, MR_i is the mean radius of the i^{th} n -round shot group, and k is the total number of groups. The mean of (70) is:

$$\langle AMR/\sigma \rangle = \langle MR/\sigma \rangle, \quad (71)$$

and its standard deviation is:

$$Std(AMR/\sigma) = \frac{1}{\sqrt{k}} Std(MR/\sigma), \quad (72)$$

and its CV is:

$$CV(AMR) = \frac{1}{\sqrt{k}} CV(MR). \quad (73)$$

In Table 16, results are presented for the means, standard deviations, and 5th and 95th percentiles of AMR/σ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$ for each of the listed n). For $n = 2$, the listed mean equals $\sqrt{\pi}/2$ via (65), and the standard deviations were computed from (69) divided by \sqrt{k} . For $n > 2$, the means were computed using (64), and the standard deviations were computed from the square root of the variance (66) and dividing by \sqrt{k} . If $n = 2$ with $k > 1$ or for $n > 2$, the percentiles are Monte Carlo estimates. A listing of percentiles of AMR/σ for the various k and n is found in Appendix 1-10, along with *MCE*'s of the estimated percentiles.

Example 19. A test agency decides to shoot 3, 5-round groups and use the Average Mean Radius as their dispersion measure. What should the requirement on AMR be to ensure that the 95% CL of σ (σ_{upper}) is the same as in Example 16.

We require:

$$AMR \leq Q,$$

where Q is the quantity that we are seeking to determine. By following the procedure outlined in Example 17:

$$Q = T_{0.05} \times \sigma_{upper}.$$

Here $T_{0.05}$ is the 5th percentile of AMR/σ for 3, 5-round groups. Substituting the required percentile (Table 16) and σ_{upper} from Example 16, we find:

$$Q = 0.855 \times 0.433 \text{ MOA} = 0.370 \text{ MOA},$$

so:

$$AMR \leq 0.370 \text{ MOA}.$$

Table 16. Table of means, standard deviations, and 5th and 95th percentiles of AMR/σ for n -round shot groups ($n=2$ -5, 10, 25) fired for a total of k groups ($k=1$ -5, 10).

k	n	Mean	Std	CV	$P\ 0.05$	$P\ 0.95$	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
1	2	0.8862	0.4633	0.5227	0.2265	1.7308	0.2556	1.9530
2			0.3276	0.3696	0.393	1.466	0.443	1.654
3			0.2675	0.3018	0.476	1.351	0.537	1.525
4			0.2316	0.2614	0.526	1.286	0.593	1.451
5			0.2072	0.2338	0.561	1.241	0.633	1.401
10			0.1465	0.1653	0.654	1.135	0.738	1.281
1	3	1.0233	0.3738	0.3653	0.459	1.683	0.448	1.645
2			0.2643	0.2583	0.611	1.479	0.597	1.446
3			0.2158	0.2109	0.684	1.394	0.669	1.362
4			0.1869	0.1826	0.728	1.343	0.712	1.313
5			0.1672	0.1634	0.757	1.307	0.740	1.278
10			0.1182	0.1155	0.833	1.222	0.814	1.194
1	4	1.0854	0.3243	0.2988	0.586	1.651	0.539	1.521
2			0.2293	0.2113	0.725	1.480	0.668	1.364
3			0.1873	0.1725	0.788	1.402	0.726	1.292
4			0.1622	0.1494	0.827	1.362	0.762	1.255
5			0.1450	0.1336	0.852	1.332	0.785	1.227
10			0.1026	0.0945	0.921	1.257	0.848	1.158
1	5	1.1210	0.2906	0.2592	0.669	1.623	0.597	1.448
2			0.2055	0.1833	0.797	1.472	0.711	1.313
3			0.1678	0.1497	0.855	1.404	0.762	1.253
4			0.1453	0.1296	0.889	1.365	0.793	1.218
5			0.1300	0.1159	0.912	1.340	0.814	1.195
10			0.0919	0.0820	0.973	1.275	0.868	1.137
1	10	1.1890	0.2063	0.1735	0.862	1.541	0.725	1.296
2			0.1459	0.1227	0.956	1.435	0.804	1.207
3			0.1191	0.1002	0.997	1.388	0.838	1.168
4			0.1031	0.0868	1.023	1.362	0.860	1.145
5			0.0923	0.0776	1.040	1.343	0.874	1.129
10			0.0652	0.0549	1.083	1.297	0.911	1.091
1	25	1.2280	0.1308	0.1065	1.018	1.449	0.829	1.180
2			0.0925	0.0753	1.079	1.382	0.878	1.126
3			0.0755	0.0615	1.105	1.354	0.900	1.103
4			0.0654	0.0533	1.121	1.336	0.913	1.088
5			0.0585	0.0476	1.132	1.325	0.922	1.079
10			0.0414	0.0337	1.160	1.296	0.945	1.056

Notes Table 16: For more details, refer to the notes at the bottom of Table 4.

Example 20. Suppose we shoot 3, 5-round groups and use the Average Mean Radius as our dispersion measure. What is σ/σ^* for a rejection rate of 0.80 with $\alpha = 0.05$? If the Average Extreme Spread is used instead, how does it compare?

Following Example 12, we seek the value of σ/σ^* for which:

$$\frac{\sigma}{\sigma^*} = \frac{T_{0.95}}{T_{0.20}}.$$

Using the percentiles of AMR/σ for 3, 5-round shot groups (Appendix 1-10):

$$\sigma/\sigma^* = 1.404/0.979 = 1.434$$

Similarly, using the percentiles of AES/σ for 3, 5-round shot groups (Appendix 1-8):

$$\sigma/\sigma^* = 3.881/2.658 = 1.460.$$

The rejection rates are nearly the same, indicating the power curves are similar as well.

Example 21. If the Average Mean Radius is used as the dispersion measure, how many k , 5-round groups need to be fired to ensure if $\sigma/\sigma^* \geq 1.5$, then the test article is rejected at least 80% of the time?

Following Example 12, choose $\alpha = 0.05$, and we know if:

$$\sigma/\sigma^* \geq T_{0.95}/T_{0.20},$$

then the test article will be rejected at least 80% of the time. Using the percentile table for multiple, 5-round shot groups (Appendix 1-10), construct the table:

k	P 0.95	P 0.20	σ/σ^*
1	1.623	0.872	1.862
2	1.472	0.945	1.557
3	1.404	0.979	1.434
4	1.365	0.998	1.369
5	1.340	1.011	1.326
10	1.275	1.043	1.222

Inspection of the table shows at least 3, 5-round groups need to be fired.

3.2.5. Radial Standard Deviation, One Shot Group

The sum of the sample variances of x and y is given by:

$$RSD^2 = \frac{1}{n} \sum_{i=1}^n [(x_i - \bar{x})^2 + (y_i - \bar{y})^2], \quad (74)$$

which is the average of the sum of squares of the distances of the impact points (x, y) from the observed center of impact (\bar{x}, \bar{y}) . Taking the square root of (74) gives the bivariate measure of dispersion known as the Radial Standard Deviation (RSD):

$$RSD = \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^n [(x_i - \bar{x})^2 + (y_i - \bar{y})^2]}. \quad (75)$$

Dividing (75) by σ :

$$RSD/\sigma = \frac{1}{\sqrt{n}} \sqrt{\sum_{i=1}^n [(x_i - \bar{x})^2/\sigma^2 + (y_i - \bar{y})^2/\sigma^2]}$$

$$RSD/\sigma = \chi(2(n-1))/\sqrt{n}, \quad (76)$$

where $\chi(2(n-1))$ is a chi random variable with $2(n-1)$ degrees of freedom.

The mean of RSD/σ is given by:

$$\langle RSD/\sigma \rangle = \langle \chi(2(n-1)) \rangle / \sqrt{n}$$

$$\langle RSD/\sigma \rangle = \sqrt{\frac{2}{n}} \frac{\Gamma((n-1) + \frac{1}{2})}{\Gamma(n-1)}, \quad (77)$$

and its variance is:

$$Var(RSD/\sigma) = \frac{1}{n} Var(\chi(2(n-1)))$$

$$= \frac{1}{n} [\langle \chi^2(2(n-1)) \rangle - \langle \chi(2(n-1)) \rangle^2]$$

$$Var(RSD/\sigma) = \frac{1}{n} \left[2(n-1) - 2 \left(\frac{\Gamma((n-1) + \frac{1}{2})}{\Gamma(n-1)} \right)^2 \right]. \quad (78)$$

The above results demonstrate the advantage of using RSD as a dispersion measure. The remarks made here about RSD parallel those made about SD in Section 3.1.5. The probability distribution of $\sqrt{n}RSD/\sigma$ is a chi distribution and analytic formulas for the first two moments are given. All percentiles and moments were computed using the built-in functions provided by Microsoft Excel 2010.

In Table 17, results are presented for the means, standard deviations, and 5th and 95th percentiles of RSD/σ for sample sizes of $n = 2-30$. The means and standard deviations were computed using (77) and the square root of the variance (78), respectively. The percentiles were determined from those of a chi-distributed random variable with $2(n-1)$ degrees of freedom scaled by $1/\sqrt{n}$ as shown in (76). A listing of percentiles for RSD/σ is found in Appendix 1-11.

Table 17. Table of means, standard deviations, and 5th and 95th percentiles for RSD/σ for sample sizes $n = 2-30$.

n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95% CL)
2	0.8862	0.4633	0.5227	0.2265	1.7308	0.2556	1.9530
3	1.0854	0.3940	0.3630	0.4867	1.7784	0.4484	1.6384
4	1.1750	0.3456	0.2941	0.6394	1.7742	0.5442	1.5100
5	1.2261	0.3110	0.2536	0.7393	1.7611	0.6030	1.4363
6	1.2592	0.2849	0.2262	0.8104	1.7468	0.6436	1.3872
7	1.2823	0.2644	0.2061	0.8640	1.7331	0.6738	1.3515
8	1.2995	0.2477	0.1906	0.9063	1.7206	0.6974	1.3241
9	1.3127	0.2338	0.1781	0.9405	1.7093	0.7165	1.3022
10	1.3231	0.2220	0.1678	0.9690	1.6991	0.7324	1.2841
11	1.3317	0.2118	0.1591	0.9932	1.6898	0.7458	1.2690
12	1.3387	0.2029	0.1516	1.0140	1.6814	0.7574	1.2560
13	1.3447	0.1951	0.1451	1.0321	1.6737	0.7676	1.2447
14	1.3497	0.1881	0.1393	1.0481	1.6666	0.7765	1.2348
15	1.3541	0.1817	0.1342	1.0623	1.6601	0.7845	1.2259
16	1.3579	0.1760	0.1296	1.0751	1.6540	0.7917	1.2180
17	1.3613	0.1708	0.1255	1.0866	1.6484	0.7982	1.2109
18	1.3643	0.1660	0.1217	1.0971	1.6432	0.8041	1.2044
19	1.3670	0.1616	0.1183	1.1066	1.6383	0.8096	1.1985
20	1.3694	0.1576	0.1151	1.1154	1.6338	0.8146	1.1931
21	1.3715	0.1538	0.1121	1.1235	1.6295	0.8192	1.1881
22	1.3735	0.1503	0.1094	1.1311	1.6254	0.8235	1.1834
23	1.3753	0.1470	0.1069	1.1380	1.6216	0.8275	1.1791
24	1.3769	0.1439	0.1045	1.1445	1.6180	0.8312	1.1751
25	1.3784	0.1410	0.1023	1.1506	1.6146	0.8347	1.1713
26	1.3798	0.1383	0.1002	1.1563	1.6113	0.8380	1.1678
27	1.3811	0.1358	0.0983	1.1617	1.6082	0.8411	1.1644
28	1.3823	0.1333	0.0964	1.1667	1.6053	0.8441	1.1613
29	1.3834	0.1310	0.0947	1.1715	1.6025	0.8468	1.1583
30	1.3845	0.1288	0.0930	1.1760	1.5998	0.8495	1.1555

Notes Table 17: For more details, refer to the notes at the bottom of Table 3.

3.2.6. Radial Standard Deviation, Multiple Shot Groups

Following a similar development as given in Section 3.1.6, a measurement of dispersion for k , n -round shot groups is constructed from the Root Mean Square (RMS) of the RSD 's of the k groups:

$$RMS-RSD = \sqrt{\frac{1}{k} \sum_{i=1}^k RSD_i^2}. \quad (79)$$

Dividing (79) by σ , and making use of (76):

$$RMS-RSD/\sigma = \frac{1}{\sqrt{kn}} \sqrt{\sum_{i=1}^k \chi_i^2 (2(n-1))}. \quad (80)$$

Note that the sum of k , chi-square independent variables each with $2(n-1)$ degrees of freedom is a chi-square variable with $2k(n-1)$ degrees of freedom; therefore:

$$\begin{aligned} RMS-RSD/\sigma &= \frac{1}{\sqrt{kn}} \sqrt{\chi^2 (2k(n-1))} \\ RMS-RSD/\sigma &= \frac{1}{\sqrt{kn}} \chi (2k(n-1)). \end{aligned} \quad (81)$$

Continuing, the mean of $RMS-RSD/\sigma$ is given by:

$$\begin{aligned} \langle RMS-RSD/\sigma \rangle &= \frac{1}{\sqrt{kn}} \langle \chi (2k(n-1)) \rangle \\ \langle RMS-RSD/\sigma \rangle &= \sqrt{\frac{2}{kn}} \frac{\Gamma(k(n-1) + \frac{1}{2})}{\Gamma(k(n-1))}, \end{aligned} \quad (82)$$

and its variance by:

$$\begin{aligned} Var(RMS-RSD/\sigma) &= \frac{1}{kn} Var(\chi (2k(n-1))) \\ Var(RMS-RSD/\sigma) &= \frac{1}{kn} \left\{ 2k(n-1) - 2 \left(\frac{\Gamma(k(n-1) + \frac{1}{2})}{\Gamma(k(n-1))} \right)^2 \right\}, \end{aligned} \quad (83)$$

and its CV is computed from:

$$CV(RMS-RSD) = \frac{\sqrt{2k(n-1) - (\chi(2k(n-1)))^2}}{\langle \chi(2k(n-1)) \rangle}. \quad (84)$$

A comparison of the CV 's given by (84) with that given for a group average over RSD 's would show that the CV of $RMS-RSD$ is slightly smaller, which means that $RMS-RSD$ is a slightly more efficient estimator of σ .

In Table 18, results are presented for the means, standard deviations, and 5th and 95th percentiles of $RMS-RSD/\sigma$ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$ for each of the listed n). The means are computed from (82), and the standard deviations from the square root of the variance (83), and the percentiles are computed from a chi random variable ($2k(n-1)$ degrees of freedom) scaled by $1/\sqrt{kn}$ in accordance with (81). A listing of additional percentiles of $RMS-RSD/\sigma$ for the various n and k is found in Appendix 1-12.

Table 18. Table of means, standard deviations, and 5th and 95th percentiles of $RMS-RSD/\sigma$ for n -round shot groups ($n = 2-5, 10, 25$) fired for a total of k groups ($k = 1-5, 10$).

k	n	Mean	Std	CV	P 0.05	P 0.95	$\hat{\sigma}/\sigma$ (5% CL)	$\hat{\sigma}/\sigma$ (95%)
1	2	0.8862	0.4633	0.5227	0.2265	1.7308	0.2556	1.9530
2		0.9400	0.3412	0.3630	0.4215	1.5401	0.4484	1.6384
3		0.9594	0.2822	0.2941	0.5221	1.4487	0.5442	1.5100
4		0.9693	0.2458	0.2536	0.5844	1.3923	0.6030	1.4363
5		0.9754	0.2207	0.2262	0.6277	1.3530	0.6436	1.3872
10		0.9876	0.1571	0.1591	0.7366	1.2532	0.7458	1.2690
1	3	1.0854	0.3940	0.3630	0.4867	1.7784	0.4484	1.6384
2		1.1193	0.2839	0.2536	0.6749	1.6077	0.6030	1.4363
3		1.1309	0.2331	0.2061	0.7620	1.5285	0.6738	1.3515
4		1.1368	0.2025	0.1781	0.8145	1.4803	0.7165	1.3022
5		1.1404	0.1814	0.1591	0.8505	1.4471	0.7458	1.2690
10		1.1475	0.1287	0.1121	0.9400	1.3633	0.8192	1.1881
1	4	1.1750	0.3456	0.2941	0.6394	1.7742	0.5442	1.5100
2		1.1995	0.2473	0.2061	0.8082	1.6212	0.6738	1.3515
3		1.2079	0.2027	0.1678	0.8846	1.5511	0.7324	1.2841
4		1.2121	0.1758	0.1451	0.9303	1.5086	0.7676	1.2447
5		1.2146	0.1574	0.1296	0.9616	1.4794	0.7917	1.2180
10		1.2197	0.1116	0.0915	1.0391	1.4061	0.8520	1.1528
1	5	1.2261	0.3110	0.2536	0.7393	1.7611	0.6030	1.4363
2		1.2453	0.2218	0.1781	0.8923	1.6216	0.7165	1.3022
3		1.2518	0.1816	0.1451	0.9608	1.5581	0.7676	1.2447
4		1.2551	0.1575	0.1255	1.0018	1.5198	0.7982	1.2109
5		1.2570	0.1410	0.1121	1.0297	1.4934	0.8192	1.1881
10		1.2610	0.0998	0.0792	1.0990	1.4274	0.8716	1.1320
1	10	1.3231	0.2220	0.1678	0.9690	1.6991	0.7324	1.2841
2		1.3324	0.1576	0.1183	1.0786	1.5968	0.8096	1.1985
3		1.3354	0.1288	0.0964	1.1272	1.5508	0.8441	1.1613
4		1.3370	0.1116	0.0835	1.1561	1.5232	0.8647	1.1393
5		1.3379	0.0999	0.0746	1.1758	1.5043	0.8788	1.1244
10		1.3398	0.0707	0.0527	1.2246	1.4571	0.9140	1.0875
1	25	1.3784	0.1410	0.1023	1.1506	1.6146	0.8347	1.1713
2		1.3820	0.0999	0.0723	1.2198	1.5484	0.8826	1.1203
3		1.3832	0.0816	0.0590	1.2504	1.5188	0.9040	1.0980
4		1.3838	0.0707	0.0511	1.2686	1.5011	0.9168	1.0847
5		1.3842	0.0632	0.0457	1.2810	1.4890	0.9255	1.0757
10		1.3849	0.0447	0.0323	1.3118	1.4589	0.9472	1.0534

Notes Table 18: For more details, refer to the notes at the bottom of Table 4.

Example 22. If the Average Extreme Spread is used as the dispersion measure, how many k , 5-round shot groups need to be fired to get at least the same CV of $RMS-RSD$ for 10, 5-round shot groups?

Equating CV 's gives:

$$CV(AES(k, 5)) = CV(RMS-RSD(10, 5))$$

$$CV(ES(5))/\sqrt{k} = CV(RMS-RSD(10, 5))$$

(See Example 5 for an explanation of the notation.) Solving for k and substituting for the CV 's (Tables 11 and 18):

$$k = (0.270/0.0792)^2 = 11.6,$$

which shows at least 12, 5-round groups need to be fired at a cost of expending 20% more rounds.

Example 23. Compare the power curves ($\alpha = 0.05$) of $RMS-SD$ and $RMS-RSD$ for 3, 5-round shot groups.

Following the procedure outlined in Example 10, the power curves of $RMS-SD$ and $RMS-RSD$ are generated (Figure 6). The value of σ/σ^* at a rejection rate of 0.80 is 13.5% smaller for $RMS-RSD$, which reflects the improved power of an acceptance test for which the bivariate measure can be used in lieu of the univariate measure.

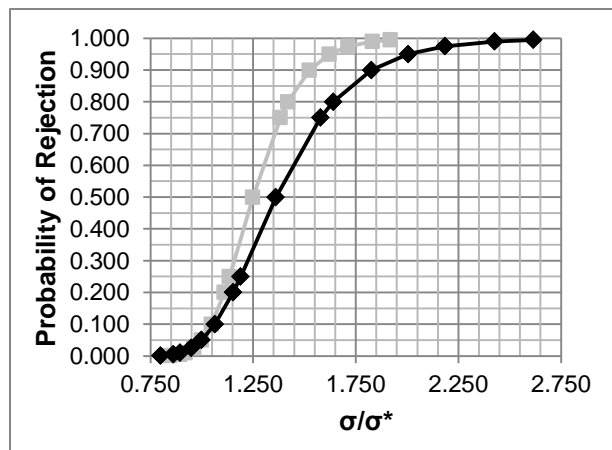


Figure 6. A comparison between the power curves ($\alpha = 0.05$) of $RMS-SD$ (black diamonds) and $RMS-RSD$ (gray squares) for 3, 5-round groups.

4. Conclusions

In this report, we have examined the univariate dispersion measures (Section 3.1): Range (R), Mean Deviation (MD), and Standard Deviation (SD); and bivariate measures (Section 3.2): Extreme Spread (ES), Mean Radius (MR), and Radial Standard Deviation (RSD). First, an analysis of each dispersion measure was given as applied to one, n -round shot group ($n = 2-30$). Percentiles and the first two moments for these measures were computed in units on the population standard deviation, σ . The probability distributions of $\sqrt{n}SD/\sigma$ and $\sqrt{n}RSD/\sigma$ are chi random variables and their percentiles are easily computed by taking the square root of the percentiles from the corresponding chi-square

distribution. Conversely, for the other dispersion measures, Monte Carlo simulations were used to estimate percentiles (except for the case with $n = 2$ as discussed in the text).

Next, we extended our analysis to treat various k multiples of n -round shot groups, where the multiples are independent, repeat firings of n -round groups from the same population. We examined the *RMS* of the *SD*'s and similarly for the *RSD*'s. These dispersion measures were denoted as *RMS-SD* and *RMS-RSD*, respectively. We note that $\sqrt{kn} \text{ RMS-SD}/\sigma$ and $\sqrt{kn} \text{ RMS-RSD}/\sigma$ are chi distributed, and their percentiles are straightforward to compute from the chi-square distribution. We also examined the Average Range (*AR*), Average Mean Deviation (*AMD*), Average Extreme Spread (*AES*) and the Average Mean Radius (*AMR*). These dispersion measures were constructed by averaging the dispersion measure of the n -round group over all k groups. For these cases, more effort was required to compute percentiles, and Monte Carlo simulations were used to estimate the percentiles of the dispersion measures given by averages over groups. From the central limit theorem, the probability distribution of the average over a dispersion measure is asymptotically normal as the number of groups averaged over increases. The convergence to a normal distribution is rapid, and for many applications using a group average to measure dispersion, the normal-distributed approximation will be sufficient. Our Monte Carlo estimates of percentiles can be used to determine the error in using this approximation to justify that it is within one's error budget (see remarks in Section 3.1.2). Inclusion of percentile tables for dispersion measures treating averages over multiple groups is instructive, but the predominant utility in these results is for cases involving a few groups with a small sample size per group.

Finally, we provide some thoughts on choosing a dispersion measure. A metric for selecting a dispersion measure is the Coefficient of Variation, since measures with smaller *CV* provide a more precise estimate of σ . It is surprising that *SD* and *RSD* are generally not used to measure dispersion in shot groups (especially considering the ease with which the percentiles of *RMS-SD* and *RMS-RSD* can be computed). For $n > 2$, both *SD* and *RSD* have the best performance in terms of Coefficient of Variation¹—keep in mind we are comparing *SD* with other univariate measures with the same sample size and accordingly, *RSD* with other bivariate measures. When measuring the dispersion on paper targets, one cannot match the simplicity of the univariate Range or *ES*; for example, the *ES* is easily obtained by measuring the largest distance between two shots. However, with the advent of electronic targeting systems, computation of any of the dispersion measures poses no difficulty. The danger in using the Range or *ES* is failing to reject a bad round, an outlier which contributes to the computation of Range or *ES*. In this regard, the other dispersion measures are more forgiving. In addition, Grubbs suggests using the Relative Efficiency¹ as a metric to compare two dispersion measures, where the Relative Efficiency (*RE*) is defined as the square of the ratio of Coefficients of Variation, where the ratio is formed by the smaller *CV* divided by the larger *CV* so that the $RE \leq 1$. In Example 5, we have already encountered this ratio and put it to use. Considerations of *CV* and *RE* are useful tools for justifying the choice of dispersion measure. Regardless of which dispersion measure is selected, the aim of the measurement is to obtain an estimate of the unknown, population standard deviation, σ to within the precision required.

In closing, we hope the results presented in this report will prove useful to those developing requirements or tests on firearms precision. It is with this purpose that this report was prepared.

Appendix 1. Percentile Tables

All percentiles are reported in units of the population standard deviation, σ . Percentiles reported with 3 or less decimal places are Monte Carlo estimates. Monte Carlo simulations were repeated for a total of 9 runs, and the reported percentiles are the sample average over the 9 runs. The *MCE*'s were computed for each estimated percentile, and *MCE*'s are reported in accompanying tables. The standard error of each estimated percentile is one third the *MCE*, since 9 runs were averaged over. Consult Section 2 for additional details.

Appendix 1-1. Range, One Shot Group

Table A1-1.1. Percentiles of the Range (*R*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
2	4.6535	3.9697	3.6428	3.1698	2.7718	2.3262	1.8124	1.6268
3	5.08	4.44	4.12	3.685	3.317	2.907	2.427	2.252
4	5.30	4.69	4.39	3.976	3.625	3.238	2.783	2.620
5	5.47	4.86	4.59	4.190	3.852	3.473	3.039	2.873
6	5.61	5.04	4.76	4.367	4.032	3.658	3.231	3.074
7	5.71	5.15	4.89	4.496	4.172	3.808	3.387	3.234
8	5.82	5.25	4.985	4.605	4.286	3.930	3.519	3.368
9	5.91	5.34	5.07	4.698	4.384	4.038	3.636	3.486
10	5.96	5.41	5.16	4.785	4.475	4.131	3.731	3.584
11	6.03	5.475	5.229	4.861	4.554	4.213	3.822	3.676
12	6.12	5.56	5.30	4.934	4.624	4.287	3.896	3.750
13	6.16	5.59	5.350	4.989	4.686	4.349	3.967	3.827
14	6.22	5.64	5.397	5.037	4.742	4.414	4.029	3.891
15	6.21	5.70	5.45	5.094	4.795	4.466	4.090	3.952
16	6.24	5.72	5.48	5.137	4.845	4.523	4.146	4.009
17	6.36	5.80	5.56	5.194	4.898	4.570	4.194	4.058
18	6.29	5.81	5.57	5.229	4.939	4.617	4.243	4.109
19	6.36	5.87	5.618	5.264	4.971	4.651	4.285	4.150
20	6.36	5.88	5.64	5.300	5.015	4.695	4.329	4.195
21	6.44	5.92	5.688	5.344	5.048	4.732	4.367	4.234
22	6.45	5.95	5.71	5.375	5.088	4.771	4.410	4.278
23	6.50	5.98	5.729	5.384	5.107	4.801	4.444	4.314
24	6.51	6.00	5.76	5.421	5.143	4.833	4.475	4.343
25	6.57	6.03	5.79	5.457	5.179	4.865	4.506	4.377
26	6.57	6.07	5.82	5.479	5.202	4.893	4.541	4.409
27	6.59	6.086	5.852	5.509	5.228	4.923	4.569	4.438
28	6.65	6.13	5.88	5.532	5.254	4.947	4.595	4.465
29	6.58	6.13	5.89	5.555	5.275	4.969	4.622	4.493
30	6.62	6.13	5.911	5.569	5.300	4.994	4.648	4.520

Table A1-1.1. Percentiles of the Range (*R*). (CONTINUED)

n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2	0.9539	0.4506	0.3583	0.1777	0.0887	0.0443	0.0177	0.0089
3	1.590	1.026	0.902	0.619	0.431	0.306	0.192	0.134
4	1.980	1.416	1.285	0.980	0.761	0.598	0.434	0.342
5	2.257	1.699	1.572	1.261	1.032	0.849	0.667	0.560
6	2.473	1.924	1.798	1.487	1.253	1.064	0.873	0.751
7	2.643	2.109	1.985	1.676	1.443	1.254	1.054	0.927
8	2.791	2.264	2.143	1.840	1.602	1.415	1.218	1.090
9	2.914	2.397	2.276	1.975	1.739	1.550	1.344	1.214
10	3.026	2.517	2.399	2.095	1.866	1.674	1.467	1.334
11	3.125	2.618	2.501	2.204	1.975	1.784	1.577	1.443
12	3.208	2.711	2.593	2.298	2.069	1.885	1.673	1.540
13	3.283	2.787	2.673	2.387	2.164	1.976	1.767	1.630
14	3.358	2.868	2.754	2.468	2.241	2.061	1.862	1.72
15	3.424	2.942	2.829	2.543	2.323	2.137	1.931	1.796
16	3.479	3.003	2.891	2.609	2.386	2.201	2.007	1.876
17	3.539	3.064	2.951	2.669	2.450	2.273	2.074	1.947
18	3.593	3.119	3.007	2.733	2.517	2.334	2.138	2.019
19	3.639	3.174	3.063	2.786	2.571	2.398	2.202	2.069
20	3.686	3.221	3.111	2.836	2.622	2.444	2.245	2.130
21	3.727	3.264	3.155	2.882	2.669	2.490	2.296	2.173
22	3.773	3.315	3.207	2.936	2.724	2.547	2.356	2.221
23	3.812	3.355	3.246	2.980	2.769	2.595	2.401	2.271
24	3.847	3.394	3.289	3.024	2.813	2.644	2.452	2.325
25	3.878	3.427	3.321	3.059	2.853	2.682	2.495	2.369
26	3.914	3.465	3.362	3.096	2.887	2.717	2.530	2.405
27	3.952	3.506	3.401	3.138	2.932	2.763	2.573	2.453
28	3.975	3.534	3.430	3.169	2.964	2.798	2.609	2.488
29	4.010	3.570	3.466	3.205	3.002	2.833	2.650	2.524
30	4.037	3.599	3.496	3.236	3.035	2.868	2.685	2.564

Table A1-1.2. Monte Carlo Errors of the Percentiles of the Range (R).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2																
3	0.07	0.04	0.03	0.017	0.015	0.013	0.005	0.006	0.005	0.007	0.007	0.006	0.008	0.007	0.007	0.010
4	0.12	0.05	0.03	0.018	0.019	0.018	0.011	0.010	0.007	0.006	0.005	0.007	0.007	0.006	0.012	0.013
5	0.10	0.04	0.03	0.012	0.009	0.016	0.013	0.012	0.009	0.008	0.008	0.009	0.010	0.015	0.02	0.018
6	0.07	0.05	0.04	0.02	0.013	0.011	0.008	0.006	0.006	0.005	0.006	0.008	0.011	0.013	0.014	0.016
7	0.04	0.03	0.03	0.02	0.017	0.014	0.010	0.008	0.009	0.007	0.008	0.008	0.010	0.013	0.016	0.018
8	0.08	0.04	0.02	0.02	0.009	0.008	0.006	0.006	0.008	0.010	0.010	0.013	0.012	0.015	0.013	0.015
9	0.10	0.04	0.03	0.02	0.013	0.011	0.012	0.010	0.009	0.008	0.009	0.008	0.009	0.012	0.019	0.02
10	0.05	0.04	0.03	0.011	0.009	0.008	0.007	0.008	0.006	0.006	0.008	0.014	0.014	0.013	0.015	0.02
11	0.07	0.02	0.02	0.018	0.014	0.010	0.006	0.006	0.006	0.007	0.004	0.007	0.006	0.010	0.013	0.017
12	0.08	0.04	0.03	0.019	0.017	0.012	0.011	0.012	0.003	0.007	0.005	0.010	0.013	0.014	0.016	0.017
13	0.08	0.04	0.02	0.012	0.012	0.008	0.011	0.010	0.007	0.005	0.005	0.004	0.006	0.007	0.011	0.009
14	0.05	0.04	0.02	0.02	0.02	0.009	0.008	0.008	0.003	0.005	0.006	0.010	0.008	0.010	0.02	0.03
15	0.06	0.04	0.03	0.017	0.014	0.010	0.008	0.006	0.008	0.006	0.007	0.006	0.008	0.016	0.010	0.019
16	0.04	0.03	0.04	0.02	0.015	0.010	0.008	0.007	0.008	0.007	0.008	0.008	0.008	0.014	0.019	0.013
17	0.09	0.04	0.03	0.018	0.013	0.010	0.007	0.006	0.006	0.007	0.009	0.013	0.012	0.013	0.016	0.015
18	0.06	0.04	0.03	0.02	0.014	0.009	0.008	0.009	0.005	0.009	0.010	0.011	0.014	0.013	0.017	0.019
19	0.06	0.04	0.02	0.018	0.016	0.010	0.007	0.009	0.008	0.007	0.006	0.006	0.008	0.013	0.014	0.02
20	0.05	0.05	0.03	0.015	0.016	0.012	0.006	0.008	0.007	0.005	0.006	0.007	0.011	0.010	0.017	0.016
21	0.06	0.03	0.02	0.010	0.012	0.012	0.008	0.006	0.005	0.005	0.003	0.003	0.006	0.006	0.013	0.02
22	0.08	0.03	0.03	0.008	0.008	0.010	0.006	0.005	0.006	0.004	0.005	0.008	0.008	0.009	0.015	0.02
23	0.05	0.04	0.02	0.02	0.010	0.008	0.005	0.006	0.005	0.005	0.005	0.007	0.008	0.007	0.009	0.015
24	0.04	0.03	0.03	0.017	0.013	0.012	0.010	0.009	0.007	0.007	0.007	0.006	0.002	0.008	0.009	0.017
25	0.08	0.04	0.04	0.02	0.007	0.009	0.009	0.009	0.007	0.004	0.006	0.005	0.009	0.011	0.015	0.013
26	0.06	0.04	0.03	0.018	0.012	0.008	0.009	0.008	0.007	0.005	0.006	0.005	0.008	0.009	0.015	0.018
27	0.05	0.02	0.02	0.011	0.007	0.008	0.005	0.005	0.005	0.008	0.007	0.009	0.009	0.009	0.011	0.014
28	0.08	0.04	0.03	0.014	0.012	0.008	0.007	0.006	0.005	0.006	0.007	0.007	0.008	0.013	0.016	0.02
29	0.07	0.03	0.04	0.02	0.014	0.009	0.006	0.007	0.006	0.006	0.006	0.006	0.004	0.010	0.010	0.018
30	0.06	0.03	0.02	0.012	0.012	0.009	0.008	0.007	0.006	0.005	0.006	0.006	0.007	0.009	0.012	0.014

Appendix 1-2. Average Range, Multiple Shot Groups

Table A1-2.1. Percentiles of the Average Range (AR).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
1	2	4.6535	3.9697	3.6428	3.1698	2.7718	2.3262	1.8124	1.6268
2		3.44	3.017	2.802	2.499	2.239	1.947	1.615	1.495
3		2.97	2.622	2.457	2.221	2.015	1.791	1.531	1.436
4		2.70	2.408	2.264	2.061	1.892	1.697	1.477	1.395
5		2.51	2.253	2.132	1.950	1.804	1.636	1.444	1.372
10		2.070	1.899	1.815	1.695	1.595	1.483	1.352	1.303
1	3	5.08	4.44	4.12	3.685	3.317	2.907	2.427	2.252
2		3.96	3.539	3.343	3.048	2.809	2.534	2.215	2.096
3		3.49	3.159	3.008	2.779	2.586	2.370	2.116	2.021
4		3.26	2.958	2.826	2.628	2.464	2.276	2.061	1.979
5		3.07	2.804	2.690	2.515	2.374	2.212	2.020	1.949
10		2.639	2.470	2.388	2.270	2.171	2.059	1.927	1.877
1	4	5.30	4.69	4.39	3.976	3.625	3.238	2.783	2.620
2		4.27	3.859	3.66	3.382	3.146	2.882	2.573	2.460
3		3.82	3.501	3.344	3.123	2.939	2.725	2.479	2.389
4		3.55	3.285	3.154	2.968	2.809	2.633	2.425	2.346
5		3.39	3.145	3.032	2.871	2.734	2.574	2.386	2.317
10		2.975	2.815	2.738	2.624	2.529	2.419	2.291	2.242
1	5	5.47	4.86	4.59	4.190	3.852	3.473	3.039	2.873
2		4.46	4.079	3.889	3.615	3.384	3.132	2.833	2.724
3		4.04	3.719	3.575	3.361	3.184	2.979	2.741	2.650
4		3.79	3.53	3.403	3.223	3.069	2.893	2.685	2.609
5		3.63	3.390	3.280	3.119	2.982	2.828	2.647	2.579
10		3.222	3.071	2.994	2.881	2.787	2.681	2.553	2.506
1	10	5.96	5.41	5.16	4.785	4.475	4.131	3.731	3.584
2		5.00	4.67	4.500	4.253	4.042	3.811	3.541	3.44
3		4.636	4.349	4.222	4.033	3.869	3.681	3.460	3.378
4		4.41	4.176	4.060	3.893	3.754	3.595	3.409	3.339
5		4.249	4.051	3.950	3.804	3.681	3.541	3.375	3.314
10		3.897	3.756	3.689	3.585	3.499	3.404	3.287	3.244
1	25	6.57	6.03	5.79	5.457	5.179	4.865	4.506	4.377
2		5.67	5.36	5.204	4.983	4.793	4.582	4.341	4.252
3		5.335	5.077	4.954	4.776	4.628	4.464	4.269	4.197
4		5.11	4.903	4.804	4.658	4.531	4.389	4.221	4.159
5		5.00	4.807	4.708	4.579	4.467	4.340	4.193	4.137
10		4.666	4.538	4.479	4.386	4.309	4.221	4.118	4.080

Table A1-2.1. Percentiles of the Average Range (AR). (CONTINUED)

k	n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2	0.9539	0.4506	0.3583	0.1777	0.0887	0.0443	0.0177	0.0089
2		1.048	0.673	0.592	0.408	0.286	0.198	0.127	0.091
3		1.078	0.765	0.696	0.529	0.411	0.321	0.234	0.185
4		1.091	0.820	0.758	0.605	0.494	0.408	0.319	0.268
5		1.100	0.853	0.797	0.658	0.554	0.472	0.387	0.334
10		1.115	0.939	0.898	0.792	0.709	0.642	0.568	0.517
1	3	1.590	1.026	0.902	0.619	0.431	0.306	0.192	0.134
2		1.646	1.241	1.149	0.920	0.754	0.620	0.483	0.400
3		1.661	1.328	1.249	1.056	0.907	0.788	0.660	0.580
4		1.668	1.378	1.309	1.137	1.004	0.898	0.781	0.709
5		1.672	1.412	1.352	1.197	1.073	0.971	0.865	0.791
10		1.683	1.498	1.453	1.339	1.249	1.172	1.089	1.030
1	4	1.980	1.416	1.285	0.980	0.761	0.598	0.434	0.342
2		2.017	1.615	1.522	1.287	1.109	0.962	0.804	0.711
3		2.033	1.701	1.623	1.424	1.269	1.146	1.007	0.917
4		2.041	1.750	1.682	1.506	1.369	1.253	1.124	1.041
5		2.045	1.786	1.724	1.566	1.442	1.336	1.220	1.144
10		2.051	1.866	1.821	1.707	1.614	1.536	1.448	1.393
1	5	2.257	1.699	1.572	1.261	1.032	0.849	0.667	0.560
2		2.296	1.896	1.803	1.564	1.379	1.229	1.069	0.964
3		2.302	1.975	1.897	1.701	1.545	1.416	1.273	1.181
4		2.309	2.024	1.956	1.780	1.642	1.525	1.398	1.311
5		2.312	2.058	1.998	1.840	1.715	1.613	1.496	1.421
10		2.318	2.138	2.095	1.981	1.890	1.814	1.724	1.665
1	10	3.026	2.517	2.399	2.095	1.866	1.674	1.467	1.334
2		3.052	2.684	2.597	2.374	2.199	2.047	1.886	1.77
3		3.061	2.757	2.684	2.499	2.349	2.225	2.087	1.991
4		3.064	2.802	2.739	2.575	2.444	2.336	2.209	2.123
5		3.067	2.832	2.774	2.627	2.509	2.409	2.295	2.222
10		3.072	2.904	2.863	2.758	2.672	2.599	2.516	2.459
1	25	3.878	3.427	3.321	3.059	2.853	2.682	2.495	2.369
2		3.906	3.582	3.505	3.307	3.150	3.019	2.873	2.777
3		3.913	3.646	3.581	3.417	3.285	3.178	3.055	2.972
4		3.916	3.684	3.628	3.484	3.369	3.268	3.155	3.082
5		3.920	3.713	3.663	3.533	3.429	3.339	3.239	3.171
10		3.926	3.776	3.739	3.646	3.571	3.506	3.430	3.379

Table A1-2.2. Monte Carlo Errors of the Percentiles of the Average Range (*AR*).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2																
2	2	0.04	0.016	0.018	0.011	0.012	0.003	0.006	0.007	0.005	0.004	0.005	0.004	0.005	0.006	0.005	0.005
3	2	0.04	0.019	0.012	0.008	0.007	0.005	0.004	0.006	0.004	0.004	0.003	0.003	0.003	0.005	0.007	0.006
4	2	0.03	0.02	0.012	0.009	0.010	0.007	0.004	0.005	0.004	0.003	0.004	0.005	0.005	0.006	0.007	0.007
5	2	0.03	0.013	0.012	0.007	0.005	0.005	0.005	0.005	0.003	0.003	0.004	0.003	0.004	0.004	0.006	0.004
10	2	0.02	0.016	0.007	0.003	0.004	0.003	0.003	0.003	0.0012	0.0018	0.002	0.0017	0.003	0.003	0.006	0.009
1	3	0.07	0.04	0.03	0.017	0.015	0.013	0.005	0.006	0.005	0.007	0.007	0.006	0.008	0.007	0.007	0.010
2	3	0.06	0.02	0.019	0.017	0.015	0.010	0.006	0.007	0.005	0.004	0.005	0.006	0.005	0.006	0.007	0.011
3	3	0.05	0.014	0.011	0.011	0.010	0.005	0.007	0.006	0.005	0.005	0.006	0.005	0.003	0.008	0.009	0.011
4	3	0.03	0.02	0.016	0.008	0.009	0.008	0.006	0.006	0.004	0.003	0.002	0.004	0.003	0.005	0.006	0.008
5	3	0.03	0.012	0.013	0.009	0.007	0.006	0.004	0.003	0.002	0.004	0.004	0.005	0.005	0.006	0.009	0.015
10	3	0.014	0.015	0.011	0.006	0.004	0.003	0.0019	0.003	0.003	0.004	0.003	0.003	0.005	0.006	0.008	0.009
1	4	0.12	0.05	0.03	0.018	0.019	0.018	0.011	0.010	0.007	0.006	0.005	0.007	0.007	0.006	0.012	0.013
2	4	0.06	0.02	0.03	0.013	0.008	0.007	0.005	0.004	0.005	0.003	0.004	0.006	0.007	0.007	0.008	0.007
3	4	0.05	0.018	0.02	0.013	0.007	0.009	0.006	0.006	0.005	0.002	0.003	0.004	0.003	0.008	0.010	0.011
4	4	0.03	0.02	0.019	0.007	0.010	0.008	0.006	0.005	0.003	0.003	0.003	0.004	0.003	0.004	0.008	0.008
5	4	0.05	0.018	0.011	0.010	0.009	0.006	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.007	0.008	0.011
10	4	0.02	0.013	0.009	0.005	0.003	0.004	0.003	0.003	0.002	0.003	0.004	0.003	0.004	0.004	0.007	0.007
1	5	0.10	0.04	0.03	0.012	0.009	0.016	0.013	0.012	0.009	0.008	0.008	0.009	0.010	0.015	0.02	0.018
2	5	0.03	0.02	0.013	0.014	0.010	0.007	0.007	0.006	0.006	0.007	0.006	0.007	0.007	0.007	0.013	0.015
3	5	0.05	0.019	0.019	0.012	0.008	0.008	0.005	0.004	0.003	0.003	0.002	0.004	0.004	0.005	0.004	0.008
4	5	0.03	0.03	0.02	0.009	0.005	0.006	0.007	0.005	0.004	0.004	0.005	0.005	0.004	0.006	0.005	0.008
5	5	0.03	0.013	0.010	0.008	0.005	0.003	0.002	0.002	0.003	0.002	0.003	0.004	0.003	0.004	0.010	0.010
10	5	0.013	0.007	0.006	0.005	0.005	0.003	0.002	0.002	0.002	0.0014	0.0014	0.004	0.003	0.002	0.008	0.009
1	10	0.05	0.04	0.03	0.011	0.009	0.008	0.007	0.008	0.006	0.006	0.008	0.014	0.014	0.013	0.015	0.02
2	10	0.03	0.03	0.019	0.010	0.008	0.005	0.006	0.004	0.003	0.005	0.006	0.008	0.010	0.013	0.015	0.03
3	10	0.02	0.010	0.012	0.007	0.008	0.007	0.006	0.005	0.004	0.003	0.004	0.004	0.006	0.006	0.008	0.016
4	10	0.03	0.02	0.015	0.010	0.008	0.006	0.007	0.005	0.006	0.006	0.006	0.007	0.005	0.004	0.008	0.007
5	10	0.02	0.016	0.012	0.008	0.007	0.006	0.005	0.003	0.004	0.005	0.005	0.007	0.007	0.006	0.009	0.009
10	10	0.02	0.009	0.009	0.004	0.004	0.004	0.003	0.003	0.0016	0.003	0.003	0.004	0.004	0.004	0.006	0.009
1	25	0.08	0.04	0.04	0.02	0.007	0.009	0.009	0.009	0.007	0.004	0.006	0.005	0.009	0.011	0.015	0.013
2	25	0.04	0.03	0.016	0.011	0.009	0.004	0.003	0.004	0.005	0.004	0.004	0.004	0.004	0.005	0.007	0.013
3	25	0.02	0.014	0.009	0.010	0.007	0.007	0.007	0.006	0.004	0.002	0.003	0.004	0.004	0.007	0.010	0.009
4	25	0.03	0.015	0.012	0.010	0.008	0.007	0.004	0.003	0.003	0.003	0.003	0.004	0.004	0.005	0.004	0.006
5	25	0.04	0.014	0.005	0.009	0.008	0.004	0.0015	0.002	0.0018	0.003	0.003	0.003	0.004	0.005	0.008	0.006
10	25	0.02	0.010	0.008	0.005	0.003	0.003	0.002	0.003	0.002	0.0019	0.002	0.004	0.004	0.003	0.007	0.007

Appendix 1-3. Mean Deviation, One Shot group

Table A1-3.1. Percentiles of the Mean Deviation (*MD*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
2	2.3268	1.9849	1.8214	1.5849	1.3859	1.1631	0.9062	0.8134
3	1.93	1.702	1.583	1.417	1.275	1.116	0.931	0.865
4	1.80	1.586	1.487	1.342	1.221	1.087	0.932	0.876
5	1.694	1.500	1.416	1.290	1.185	1.068	0.932	0.882
6	1.618	1.445	1.369	1.252	1.157	1.051	0.929	0.884
7	1.544	1.399	1.327	1.220	1.133	1.037	0.924	0.882
8	1.511	1.364	1.294	1.195	1.115	1.025	0.920	0.881
9	1.46	1.327	1.267	1.175	1.100	1.016	0.916	0.881
10	1.426	1.302	1.240	1.157	1.087	1.008	0.913	0.879
11	1.389	1.270	1.217	1.137	1.074	0.998	0.910	0.877
12	1.367	1.256	1.202	1.125	1.063	0.991	0.908	0.877
13	1.342	1.241	1.189	1.116	1.054	0.986	0.904	0.874
14	1.316	1.219	1.172	1.104	1.045	0.979	0.901	0.873
15	1.300	1.202	1.159	1.092	1.036	0.973	0.899	0.871
16	1.286	1.194	1.150	1.084	1.030	0.969	0.897	0.870
17	1.272	1.183	1.141	1.077	1.024	0.964	0.895	0.869
18	1.260	1.172	1.129	1.068	1.018	0.961	0.893	0.868
19	1.244	1.158	1.119	1.060	1.012	0.957	0.892	0.868
20	1.230	1.148	1.110	1.055	1.007	0.952	0.889	0.865
21	1.219	1.140	1.104	1.049	1.003	0.949	0.887	0.864
22	1.209	1.134	1.098	1.044	0.998	0.947	0.886	0.864
23	1.207	1.126	1.091	1.038	0.993	0.944	0.885	0.862
24	1.190	1.119	1.084	1.033	0.989	0.941	0.883	0.862
25	1.186	1.116	1.080	1.030	0.987	0.939	0.882	0.860
26	1.179	1.109	1.075	1.025	0.983	0.936	0.881	0.860
27	1.175	1.102	1.069	1.019	0.978	0.933	0.879	0.859
28	1.163	1.096	1.062	1.016	0.977	0.932	0.878	0.858
29	1.158	1.089	1.059	1.013	0.975	0.930	0.877	0.857
30	1.149	1.087	1.055	1.009	0.970	0.927	0.876	0.857

Table A1-3.1. Percentiles of the Mean Deviation (*MD*). (CONTINUED)

n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2	0.4769	0.2253	0.1791	0.0889	0.0443	0.0222	0.0089	0.0044
3	0.613	0.396	0.348	0.238	0.165	0.116	0.073	0.052
4	0.662	0.473	0.431	0.329	0.256	0.200	0.145	0.114
5	0.691	0.521	0.482	0.387	0.316	0.262	0.205	0.171
6	0.710	0.554	0.517	0.427	0.360	0.305	0.250	0.217
7	0.723	0.578	0.543	0.459	0.394	0.342	0.286	0.252
8	0.733	0.596	0.564	0.484	0.423	0.373	0.319	0.284
9	0.741	0.611	0.581	0.504	0.446	0.398	0.344	0.310
10	0.746	0.623	0.594	0.521	0.464	0.416	0.366	0.331
11	0.751	0.634	0.606	0.536	0.481	0.436	0.385	0.352
12	0.755	0.642	0.615	0.547	0.495	0.450	0.402	0.372
13	0.758	0.649	0.624	0.559	0.507	0.464	0.418	0.388
14	0.761	0.657	0.632	0.568	0.519	0.477	0.431	0.403
15	0.764	0.662	0.638	0.578	0.529	0.488	0.444	0.414
16	0.766	0.668	0.644	0.585	0.537	0.498	0.453	0.424
17	0.768	0.673	0.650	0.591	0.545	0.507	0.464	0.434
18	0.769	0.676	0.654	0.598	0.552	0.515	0.473	0.443
19	0.772	0.681	0.659	0.604	0.560	0.522	0.481	0.455
20	0.772	0.685	0.663	0.609	0.565	0.529	0.490	0.463
21	0.774	0.688	0.667	0.613	0.571	0.536	0.497	0.471
22	0.775	0.691	0.670	0.619	0.577	0.543	0.504	0.478
23	0.775	0.693	0.673	0.622	0.582	0.548	0.511	0.486
24	0.776	0.695	0.676	0.626	0.586	0.553	0.516	0.490
25	0.777	0.698	0.679	0.630	0.591	0.557	0.520	0.496
26	0.778	0.701	0.682	0.634	0.596	0.562	0.525	0.501
27	0.779	0.703	0.685	0.637	0.599	0.567	0.531	0.508
28	0.780	0.705	0.687	0.640	0.603	0.572	0.537	0.513
29	0.781	0.707	0.689	0.644	0.607	0.576	0.540	0.518
30	0.781	0.708	0.690	0.646	0.610	0.580	0.545	0.523

Table A1-3.2. Monte Carlo Errors of the Percentiles of the Mean Deviation (*MD*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2																
3	0.03	0.016	0.015	0.009	0.009	0.007	0.005	0.005	0.005	0.003	0.0018	0.002	0.003	0.003	0.003	0.003
4	0.03	0.017	0.012	0.005	0.005	0.005	0.003	0.003	0.0010	0.0017	0.0018	0.003	0.003	0.003	0.003	0.004
5	0.02	0.008	0.005	0.004	0.006	0.005	0.003	0.003	0.002	0.002	0.002	0.003	0.003	0.003	0.004	0.005
6	0.02	0.014	0.008	0.005	0.005	0.0019	0.0013	0.002	0.002	0.002	0.003	0.003	0.003	0.004	0.004	0.004
7	0.02	0.006	0.006	0.006	0.005	0.004	0.0016	0.003	0.0015	0.0017	0.002	0.003	0.004	0.004	0.005	0.006
8	0.01	0.005	0.004	0.003	0.004	0.004	0.0018	0.002	0.002	0.002	0.002	0.003	0.003	0.003	0.005	0.006
9	0.03	0.012	0.008	0.005	0.005	0.003	0.003	0.002	0.0014	0.0013	0.0017	0.002	0.002	0.002	0.005	0.005
10	0.015	0.008	0.004	0.004	0.004	0.0018	0.003	0.002	0.002	0.002	0.0019	0.0014	0.002	0.002	0.004	0.004
11	0.010	0.008	0.006	0.004	0.003	0.0019	0.0017	0.0013	0.0018	0.0016	0.002	0.002	0.002	0.003	0.004	0.003
12	0.010	0.011	0.008	0.004	0.004	0.002	0.003	0.002	0.0018	0.0015	0.0016	0.002	0.0013	0.0016	0.003	0.003
13	0.011	0.006	0.006	0.005	0.003	0.003	0.0015	0.0013	0.0016	0.0014	0.0018	0.002	0.002	0.0014	0.003	0.0015
14	0.015	0.006	0.003	0.002	0.003	0.0018	0.0018	0.002	0.0018	0.0018	0.0017	0.0014	0.0018	0.002	0.003	0.004
15	0.017	0.006	0.002	0.003	0.0013	0.002	0.002	0.002	0.0017	0.0013	0.0013	0.0012	0.0019	0.0019	0.0017	0.003
16	0.016	0.008	0.004	0.002	0.0016	0.0016	0.0012	0.0010	0.0014	0.0013	0.0016	0.0013	0.0012	0.0014	0.003	0.004
17	0.010	0.005	0.004	0.003	0.002	0.0015	0.0018	0.002	0.0012	0.0013	0.0017	0.002	0.0014	0.003	0.003	0.005
18	0.012	0.008	0.005	0.002	0.003	0.0017	0.0014	0.0013	0.0014	0.0012	0.0012	0.0013	0.0018	0.002	0.003	0.003
19	0.013	0.005	0.005	0.003	0.002	0.0016	0.0015	0.0014	0.0015	0.0010	0.0010	0.0013	0.0019	0.002	0.003	0.004
20	0.007	0.005	0.004	0.0007	0.002	0.0017	0.0013	0.0012	0.0012	0.0014	0.0011	0.0012	0.0019	0.0018	0.003	0.002
21	0.005	0.004	0.003	0.002	0.002	0.0017	0.0011	0.0011	0.0009	0.0011	0.0012	0.0012	0.0017	0.0019	0.003	0.004
22	0.011	0.004	0.004	0.002	0.0016	0.0012	0.0014	0.0014	0.0012	0.0011	0.0015	0.0015	0.0017	0.003	0.003	0.003
23	0.008	0.004	0.003	0.004	0.003	0.003	0.0018	0.0016	0.0010	0.0010	0.0010	0.0017	0.0013	0.0015	0.003	0.002
24	0.011	0.005	0.004	0.004	0.002	0.002	0.0016	0.0012	0.0013	0.0012	0.0010	0.0017	0.0019	0.0019	0.003	0.004
25	0.007	0.0013	0.002	0.002	0.002	0.002	0.0015	0.0017	0.0012	0.0009	0.0009	0.0016	0.0018	0.0014	0.0013	0.003
26	0.012	0.006	0.004	0.002	0.0015	0.0016	0.0015	0.0016	0.0008	0.0005	0.0007	0.0013	0.0010	0.0008	0.0019	0.002
27	0.008	0.004	0.003	0.0016	0.0014	0.0018	0.0015	0.0016	0.0011	0.0013	0.0013	0.0017	0.0014	0.002	0.003	0.003
28	0.004	0.003	0.004	0.003	0.002	0.0014	0.0006	0.0008	0.0011	0.0014	0.0013	0.0016	0.0013	0.0017	0.003	0.004
29	0.003	0.005	0.004	0.002	0.003	0.0016	0.0012	0.0010	0.0007	0.0012	0.0013	0.0017	0.0016	0.002	0.003	0.002
30	0.009	0.005	0.004	0.002	0.0014	0.0019	0.0016	0.0014	0.0010	0.0007	0.0007	0.0011	0.0015	0.0018	0.0017	0.0018

Appendix 1-4. Average Mean Deviation, Multiple Shot Groups

Table A1-4.1. Percentiles of the Average Mean Deviation (*AMD*).

k	n	P 0.999	P	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
1	2	2.3268	1.9849	1.8214	1.5849	1.3859	1.1631	0.9062	0.8134
2		1.72	1.508	1.401	1.246	1.120	0.976	0.811	0.750
3		1.48	1.315	1.228	1.108	1.007	0.895	0.764	0.716
4		1.347	1.204	1.133	1.030	0.943	0.848	0.737	0.697
5		1.263	1.126	1.062	0.977	0.902	0.819	0.722	0.686
10		1.042	0.949	0.907	0.849	0.798	0.742	0.676	0.652
1	3	1.93	1.702	1.583	1.417	1.275	1.116	0.931	0.865
2		1.525	1.368	1.287	1.176	1.079	0.976	0.851	0.806
3		1.340	1.222	1.164	1.074	1.000	0.914	0.815	0.780
4		1.252	1.139	1.085	1.010	0.948	0.878	0.794	0.763
5		1.180	1.085	1.039	0.972	0.915	0.853	0.778	0.751
10		1.008	0.945	0.916	0.872	0.835	0.792	0.741	0.722
1	4	1.80	1.586	1.487	1.342	1.221	1.087	0.932	0.876
2		1.438	1.299	1.235	1.140	1.060	0.970	0.866	0.827
3		1.290	1.182	1.132	1.054	0.989	0.916	0.832	0.802
4		1.204	1.110	1.066	1.002	0.947	0.886	0.815	0.788
5		1.145	1.063	1.023	0.966	0.919	0.865	0.801	0.778
10		1.002	0.948	0.921	0.882	0.850	0.813	0.770	0.753
1	5	1.694	1.500	1.416	1.290	1.185	1.068	0.932	0.882
2		1.380	1.260	1.197	1.111	1.042	0.962	0.869	0.835
3		1.247	1.148	1.101	1.032	0.976	0.914	0.841	0.813
4		1.162	1.083	1.044	0.988	0.940	0.887	0.824	0.800
5		1.111	1.041	1.009	0.957	0.916	0.868	0.813	0.792
10		0.987	0.943	0.918	0.884	0.855	0.823	0.784	0.769
1	10	1.426	1.302	1.240	1.157	1.087	1.008	0.913	0.879
2		1.212	1.129	1.090	1.033	0.985	0.932	0.868	0.845
3		1.123	1.058	1.026	0.981	0.943	0.899	0.848	0.829
4		1.067	1.014	0.988	0.949	0.917	0.880	0.836	0.820
5		1.037	0.987	0.963	0.929	0.900	0.867	0.827	0.813
10		0.949	0.916	0.900	0.877	0.857	0.834	0.807	0.796
1	25	1.186	1.116	1.080	1.030	0.987	0.939	0.882	0.860
2		1.062	1.012	0.988	0.954	0.924	0.891	0.853	0.838
3		1.009	0.968	0.949	0.922	0.898	0.872	0.840	0.828
4		0.973	0.942	0.925	0.902	0.882	0.859	0.832	0.822
5		0.956	0.926	0.911	0.890	0.872	0.851	0.827	0.818
10		0.901	0.882	0.872	0.857	0.845	0.831	0.814	0.807

Table A1-4.1. Percentiles of the Average Mean Deviation (*AMD*). (CONTINUED)

k	n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2	0.4769	0.2253	0.1791	0.0889	0.0443	0.0222	0.0089	0.0044
2		0.527	0.338	0.298	0.205	0.142	0.100	0.064	0.045
3		0.540	0.384	0.348	0.265	0.205	0.162	0.118	0.094
4		0.545	0.408	0.378	0.302	0.246	0.203	0.159	0.132
5		0.549	0.426	0.398	0.329	0.277	0.235	0.191	0.164
10		0.557	0.469	0.449	0.396	0.355	0.322	0.285	0.261
1	3	0.613	0.396	0.348	0.238	0.165	0.116	0.073	0.052
2		0.632	0.476	0.441	0.353	0.288	0.238	0.187	0.157
3		0.640	0.512	0.482	0.406	0.349	0.302	0.254	0.222
4		0.643	0.531	0.504	0.438	0.387	0.344	0.297	0.269
5		0.644	0.544	0.520	0.460	0.413	0.375	0.333	0.304
10		0.647	0.576	0.559	0.515	0.480	0.451	0.418	0.396
1	4	0.662	0.473	0.431	0.329	0.256	0.200	0.145	0.114
2		0.678	0.542	0.510	0.430	0.369	0.321	0.269	0.236
3		0.682	0.570	0.544	0.477	0.426	0.383	0.338	0.309
4		0.684	0.587	0.564	0.505	0.459	0.420	0.378	0.351
5		0.685	0.599	0.578	0.524	0.482	0.447	0.409	0.384
10		0.689	0.626	0.611	0.572	0.541	0.515	0.485	0.466
1	5	0.691	0.521	0.482	0.387	0.316	0.262	0.205	0.171
2		0.703	0.580	0.551	0.478	0.422	0.375	0.325	0.293
3		0.707	0.606	0.582	0.521	0.473	0.433	0.389	0.361
4		0.707	0.620	0.599	0.547	0.504	0.468	0.428	0.403
5		0.709	0.631	0.612	0.564	0.525	0.493	0.458	0.435
10		0.711	0.655	0.642	0.607	0.579	0.556	0.528	0.510
1	10	0.746	0.623	0.594	0.521	0.464	0.416	0.366	0.331
2		0.751	0.664	0.643	0.589	0.546	0.510	0.469	0.443
3		0.753	0.682	0.664	0.619	0.583	0.552	0.517	0.494
4		0.754	0.692	0.676	0.637	0.606	0.579	0.548	0.529
5		0.755	0.699	0.685	0.650	0.621	0.597	0.570	0.551
10		0.756	0.716	0.706	0.681	0.660	0.643	0.623	0.609
1	25	0.777	0.698	0.679	0.630	0.591	0.557	0.520	0.496
2		0.780	0.723	0.710	0.674	0.645	0.622	0.593	0.573
3		0.781	0.734	0.723	0.694	0.670	0.650	0.627	0.612
4		0.781	0.740	0.731	0.706	0.685	0.667	0.647	0.633
5		0.781	0.745	0.736	0.713	0.694	0.678	0.661	0.649
10		0.782	0.756	0.750	0.733	0.720	0.708	0.695	0.686

Table A1-4.2. Monte Carlo Errors of the Percentiles of the Average Mean Deviation (*AMD*).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2																
2	2	0.03	0.018	0.010	0.008	0.007	0.006	0.004	0.003	0.003	0.002	0.002	0.002	0.002	0.0019	0.003	0.0018
3	2	0.03	0.011	0.007	0.005	0.005	0.003	0.003	0.002	0.0018	0.003	0.003	0.002	0.003	0.003	0.003	0.003
4	2	0.02	0.010	0.009	0.003	0.003	0.003	0.002	0.002	0.0015	0.0014	0.0017	0.002	0.001	0.002	0.003	0.003
5	2	0.016	0.009	0.006	0.005	0.004	0.003	0.0016	0.0018	0.0016	0.0012	0.0010	0.002	0.002	0.0017	0.003	0.003
10	2	0.006	0.004	0.004	0.003	0.0019	0.0014	0.0016	0.0012	0.0013	0.0014	0.0012	0.0012	0.0019	0.0011	0.002	0.003
1	3	0.03	0.016	0.015	0.009	0.009	0.007	0.005	0.005	0.005	0.003	0.0018	0.002	0.003	0.003	0.003	0.003
2	3	0.02	0.010	0.008	0.006	0.006	0.005	0.0019	0.0017	0.0015	0.003	0.003	0.002	0.003	0.002	0.004	0.004
3	3	0.016	0.009	0.005	0.004	0.003	0.004	0.003	0.002	0.0011	0.0007	0.0007	0.0013	0.0013	0.0012	0.002	0.005
4	3	0.013	0.008	0.005	0.005	0.002	0.002	0.0013	0.0011	0.0009	0.0008	0.0014	0.002	0.003	0.002	0.002	0.003
5	3	0.008	0.007	0.004	0.003	0.0013	0.0016	0.0012	0.0013	0.0008	0.0015	0.0013	0.0011	0.002	0.0019	0.003	0.004
10	3	0.005	0.005	0.004	0.0019	0.0012	0.0012	0.0007	0.0005	0.0010	0.0004	0.0007	0.0011	0.0009	0.0017	0.0015	0.002
1	4	0.03	0.017	0.012	0.005	0.005	0.005	0.003	0.003	0.0010	0.0017	0.0018	0.003	0.003	0.003	0.003	0.004
2	4	0.02	0.007	0.004	0.004	0.003	0.003	0.002	0.002	0.0017	0.0005	0.0009	0.0013	0.0018	0.002	0.004	0.005
3	4	0.014	0.005	0.003	0.004	0.003	0.002	0.0016	0.0014	0.0017	0.0011	0.0012	0.0013	0.002	0.0013	0.003	0.004
4	4	0.014	0.005	0.004	0.002	0.0011	0.0015	0.0012	0.0011	0.0007	0.0013	0.0012	0.0014	0.0019	0.002	0.0019	0.004
5	4	0.011	0.007	0.005	0.002	0.002	0.0010	0.0014	0.0012	0.0013	0.0012	0.0014	0.0016	0.0014	0.0014	0.0019	0.002
10	4	0.004	0.003	0.003	0.002	0.0017	0.0011	0.0011	0.0009	0.0006	0.0013	0.0013	0.0013	0.0011	0.0012	0.0013	0.002
1	5	0.02	0.008	0.005	0.004	0.006	0.005	0.003	0.003	0.002	0.002	0.002	0.003	0.003	0.003	0.004	0.005
2	5	0.02	0.010	0.008	0.005	0.003	0.002	0.003	0.002	0.0016	0.0019	0.0016	0.002	0.003	0.002	0.003	0.003
3	5	0.008	0.005	0.004	0.004	0.002	0.0017	0.0016	0.0015	0.0014	0.0009	0.0012	0.0017	0.003	0.003	0.004	0.004
4	5	0.010	0.007	0.004	0.0016	0.003	0.002	0.0015	0.0014	0.0013	0.0016	0.0013	0.0014	0.0015	0.0014	0.003	0.004
5	5	0.011	0.004	0.003	0.0019	0.002	0.0017	0.0016	0.0014	0.0011	0.0008	0.0007	0.0012	0.0010	0.0009	0.0016	0.0017
10	5	0.006	0.002	0.0013	0.0007	0.0010	0.0014	0.0015	0.0012	0.0005	0.0006	0.0006	0.0010	0.0007	0.0015	0.0015	0.0019
1	10	0.015	0.008	0.004	0.004	0.004	0.0018	0.003	0.002	0.002	0.002	0.0019	0.0014	0.002	0.002	0.004	0.004
2	10	0.005	0.003	0.003	0.003	0.0019	0.0012	0.0018	0.0015	0.0015	0.0017	0.0019	0.0016	0.003	0.003	0.003	0.005
3	10	0.010	0.003	0.003	0.002	0.0015	0.0012	0.0008	0.0007	0.0008	0.0007	0.0008	0.0010	0.0013	0.0011	0.003	0.003
4	10	0.009	0.003	0.003	0.002	0.0012	0.0011	0.0008	0.0010	0.0010	0.0009	0.0008	0.0008	0.0013	0.0017	0.003	0.002
5	10	0.007	0.004	0.003	0.003	0.0012	0.0008	0.0009	0.0009	0.0013	0.0011	0.0013	0.0011	0.0011	0.0017	0.0018	0.0019
10	10	0.003	0.0017	0.0010	0.0010	0.0009	0.0007	0.0007	0.0006	0.0004	0.0005	0.0004	0.0004	0.0008	0.0011	0.002	0.002
1	25	0.007	0.0013	0.002	0.002	0.002	0.002	0.0015	0.0017	0.0012	0.0009	0.0009	0.0016	0.0018	0.0014	0.0013	0.003
2	25	0.006	0.0017	0.0017	0.0015	0.002	0.0010	0.0008	0.0009	0.0006	0.0008	0.0006	0.0009	0.0012	0.0017	0.003	0.003
3	25	0.006	0.002	0.0017	0.0010	0.0006	0.0006	0.0005	0.0007	0.0010	0.0009	0.0010	0.0010	0.0012	0.0015	0.0015	0.002
4	25	0.003	0.0019	0.0015	0.0011	0.0008	0.0008	0.0005	0.0005	0.0004	0.0003	0.0004	0.0005	0.0007	0.0006	0.0014	0.0012
5	25	0.005	0.002	0.0018	0.0011	0.0007	0.0007	0.0005	0.0007	0.0006	0.0008	0.0007	0.0007	0.0007	0.0009	0.0014	0.0012
10	25	0.002	0.0017	0.0010	0.0008	0.0004	0.0004	0.0003	0.0003	0.0003	0.0003	0.0004	0.0005	0.0006	0.0007	0.0011	0.0014

Appendix 1-5. Standard Deviation, One Shot Group

Table A1-5.1. Percentiles of the Standard Deviation (*SD*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
2	2.3268	1.9849	1.8214	1.5849	1.3859	1.1631	0.9062	0.8134
3	2.1460	1.8794	1.7522	1.5682	1.4132	1.2390	1.0358	0.9614
4	2.0166	1.7915	1.6841	1.5288	1.3977	1.2501	1.0772	1.0135
5	1.9218	1.7240	1.6295	1.4929	1.3775	1.2474	1.0944	1.0378
6	1.8491	1.6708	1.5857	1.4624	1.3583	1.2407	1.1022	1.0508
7	1.7912	1.6278	1.5497	1.4367	1.3412	1.2332	1.1057	1.0584
8	1.7436	1.5921	1.5197	1.4148	1.3260	1.2256	1.1070	1.0628
9	1.7037	1.5619	1.4941	1.3958	1.3126	1.2184	1.1071	1.0656
10	1.6696	1.5359	1.4719	1.3792	1.3007	1.2118	1.1064	1.0672
11	1.6401	1.5132	1.4526	1.3646	1.2901	1.2056	1.1054	1.0681
12	1.6141	1.4932	1.4354	1.3515	1.2805	1.1998	1.1042	1.0685
13	1.5911	1.4754	1.4201	1.3398	1.2718	1.1945	1.1029	1.0686
14	1.5704	1.4594	1.4063	1.3292	1.2638	1.1896	1.1015	1.0685
15	1.5518	1.4450	1.3938	1.3196	1.2566	1.1850	1.1000	1.0682
16	1.5350	1.4318	1.3824	1.3107	1.2499	1.1808	1.0986	1.0679
17	1.5195	1.4198	1.3720	1.3026	1.2437	1.1768	1.0972	1.0674
18	1.5054	1.4087	1.3624	1.2951	1.2380	1.1731	1.0958	1.0669
19	1.4923	1.3984	1.3535	1.2881	1.2327	1.1696	1.0945	1.0663
20	1.4802	1.3889	1.3452	1.2816	1.2277	1.1663	1.0932	1.0658
21	1.4690	1.3801	1.3375	1.2756	1.2230	1.1632	1.0919	1.0652
22	1.4585	1.3718	1.3303	1.2699	1.2186	1.1602	1.0907	1.0646
23	1.4487	1.3641	1.3235	1.2646	1.2145	1.1575	1.0895	1.0640
24	1.4394	1.3568	1.3172	1.2596	1.2106	1.1548	1.0884	1.0634
25	1.4308	1.3499	1.3112	1.2548	1.2069	1.1523	1.0873	1.0628
26	1.4226	1.3435	1.3055	1.2503	1.2034	1.1499	1.0862	1.0623
27	1.4149	1.3374	1.3002	1.2461	1.2001	1.1477	1.0852	1.0617
28	1.4076	1.3316	1.2951	1.2420	1.1969	1.1455	1.0842	1.0611
29	1.4006	1.3260	1.2903	1.2382	1.1939	1.1434	1.0832	1.0606
30	1.3940	1.3208	1.2857	1.2345	1.1910	1.1415	1.0823	1.0600

Table A1-5.1. Percentiles of the Standard Deviation (*SD*). (CONTINUED)

n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2	0.4769	0.2253	0.1791	0.0889	0.0443	0.0222	0.0089	0.0044
3	0.6798	0.4379	0.3857	0.2650	0.1849	0.1299	0.0819	0.0578
4	0.7691	0.5506	0.5013	0.3822	0.2966	0.2323	0.1694	0.1339
5	0.8194	0.6201	0.5742	0.4612	0.3770	0.3113	0.2438	0.2035
6	0.8516	0.6677	0.6248	0.5181	0.4369	0.3722	0.3039	0.2620
7	0.8741	0.7025	0.6623	0.5611	0.4833	0.4204	0.3530	0.3107
8	0.8906	0.7293	0.6912	0.5951	0.5205	0.4596	0.3935	0.3516
9	0.9033	0.7506	0.7144	0.6227	0.5510	0.4921	0.4277	0.3865
10	0.9134	0.7680	0.7335	0.6456	0.5766	0.5197	0.4569	0.4165
11	0.9216	0.7826	0.7495	0.6650	0.5985	0.5433	0.4822	0.4427
12	0.9283	0.7950	0.7631	0.6818	0.6174	0.5639	0.5044	0.4658
13	0.9340	0.8057	0.7750	0.6964	0.6340	0.5820	0.5241	0.4863
14	0.9388	0.8150	0.7853	0.7092	0.6487	0.5981	0.5416	0.5046
15	0.9430	0.8232	0.7945	0.7206	0.6618	0.6126	0.5574	0.5212
16	0.9467	0.8305	0.8026	0.7309	0.6737	0.6256	0.5717	0.5362
17	0.9499	0.8371	0.8099	0.7401	0.6843	0.6374	0.5847	0.5500
18	0.9527	0.8430	0.8166	0.7485	0.6941	0.6483	0.5966	0.5626
19	0.9553	0.8484	0.8226	0.7562	0.7030	0.6582	0.6076	0.5742
20	0.9575	0.8533	0.8281	0.7632	0.7112	0.6673	0.6178	0.5850
21	0.9596	0.8578	0.8332	0.7697	0.7188	0.6758	0.6272	0.5950
22	0.9615	0.8619	0.8379	0.7758	0.7259	0.6837	0.6359	0.6043
23	0.9632	0.8658	0.8422	0.7813	0.7324	0.6910	0.6441	0.6130
24	0.9647	0.8693	0.8462	0.7866	0.7385	0.6979	0.6518	0.6212
25	0.9662	0.8726	0.8500	0.7914	0.7443	0.7043	0.6590	0.6288
26	0.9675	0.8757	0.8535	0.7960	0.7497	0.7104	0.6658	0.6361
27	0.9687	0.8786	0.8568	0.8003	0.7547	0.7161	0.6721	0.6429
28	0.9698	0.8813	0.8599	0.8043	0.7595	0.7214	0.6782	0.6494
29	0.9709	0.8839	0.8628	0.8081	0.7640	0.7265	0.6839	0.6555
30	0.9719	0.8863	0.8655	0.8117	0.7683	0.7314	0.6894	0.6613

Appendix 1-6. RMS of the Standard Deviations, Multiple Shot Groups

Table A1-6.1. Percentiles of the RMS of the Standard Deviations (*RMS-SD*).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
1	2	2.3268	1.9849	1.8214	1.5849	1.3859	1.1631	0.9062	0.8134
2		1.8585	1.6276	1.5174	1.3581	1.2239	1.0730	0.8971	0.8326
3		1.6465	1.4628	1.3751	1.2482	1.1413	1.0207	0.8795	0.8275
4		1.5193	1.3629	1.2882	1.1802	1.0890	0.9861	0.8652	0.8205
5		1.4323	1.2942	1.2283	1.1328	1.0522	0.9611	0.8538	0.8140
10		1.2163	1.1222	1.0772	1.0120	0.9567	0.8941	0.8198	0.7921
1	3	2.1460	1.8794	1.7522	1.5682	1.4132	1.2390	1.0358	0.9614
2		1.7544	1.5738	1.4875	1.3628	1.2575	1.1387	0.9991	0.9474
3		1.5797	1.4356	1.3667	1.2671	1.1828	1.0875	0.9751	0.9334
4		1.4755	1.3526	1.2939	1.2088	1.1368	1.0552	0.9587	0.9228
5		1.4045	1.2958	1.2439	1.1686	1.1047	1.0324	0.9466	0.9147
10		1.2290	1.1547	1.1190	1.0672	1.0232	0.9732	0.9136	0.8912
1	4	2.0166	1.7915	1.6841	1.5288	1.3977	1.2501	1.0772	1.0135
2		1.6755	1.5226	1.4497	1.3439	1.2546	1.1535	1.0343	0.9900
3		1.5242	1.4021	1.3437	1.2591	1.1874	1.1062	1.0100	0.9742
4		1.4342	1.3299	1.2801	1.2077	1.1464	1.0767	0.9941	0.9632
5		1.3729	1.2807	1.2365	1.1724	1.1179	1.0561	0.9826	0.9551
10		1.2217	1.1584	1.1280	1.0837	1.0461	1.0032	0.9520	0.9327
1	5	1.9218	1.7240	1.6295	1.4929	1.3775	1.2474	1.0944	1.0378
2		1.6163	1.4817	1.4174	1.3242	1.2453	1.1559	1.0502	1.0109
3		1.4812	1.3735	1.3220	1.2473	1.1840	1.1120	1.0267	0.9948
4		1.4009	1.3090	1.2649	1.2009	1.1467	1.0849	1.0116	0.9841
5		1.3463	1.2649	1.2258	1.1691	1.1209	1.0661	1.0007	0.9763
10		1.2116	1.1556	1.1286	1.0894	1.0560	1.0179	0.9723	0.9552
1	10	1.6696	1.5359	1.4719	1.3792	1.3007	1.2118	1.1064	1.0672
2		1.4545	1.3630	1.3192	1.2555	1.2014	1.1399	1.0668	1.0393
3		1.3599	1.2864	1.2512	1.1999	1.1563	1.1067	1.0474	1.0252
4		1.3037	1.2408	1.2106	1.1666	1.1291	1.0864	1.0354	1.0162
5		1.2655	1.2097	1.1829	1.1438	1.1105	1.0724	1.0269	1.0098
10		1.1714	1.1327	1.1141	1.0869	1.0637	1.0371	1.0053	0.9932
1	25	1.4308	1.3499	1.3112	1.2548	1.2069	1.1523	1.0873	1.0628
2		1.2964	1.2407	1.2139	1.1749	1.1417	1.1037	1.0582	1.0411
3		1.2374	1.1925	1.1708	1.1393	1.1124	1.0816	1.0447	1.0308
4		1.2024	1.1638	1.1452	1.1180	1.0949	1.0683	1.0365	1.0244
5		1.1785	1.1442	1.1277	1.1035	1.0828	1.0592	1.0308	1.0200
10		1.1197	1.0958	1.0842	1.0673	1.0529	1.0363	1.0163	1.0087

Table A1-6.1. Percentiles of the RMS of the Standard Deviations (*RMS-SD*). (CONTINUED)

k	n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2	0.4769	0.2253	0.1791	0.0889	0.0443	0.0222	0.0089	0.0044
2		0.5887	0.3793	0.3340	0.2295	0.1601	0.1125	0.0709	0.0501
3		0.6280	0.4495	0.4093	0.3121	0.2422	0.1896	0.1383	0.1093
4		0.6478	0.4902	0.4540	0.3646	0.2981	0.2461	0.1927	0.1609
5		0.6597	0.5172	0.4840	0.4013	0.3384	0.2883	0.2354	0.2029
10		0.6834	0.5804	0.5558	0.4932	0.4439	0.4029	0.3576	0.3283
1	3	0.6798	0.4379	0.3857	0.2650	0.1849	0.1299	0.0819	0.0578
2		0.7480	0.5661	0.5242	0.4210	0.3442	0.2841	0.2225	0.1857
3		0.7709	0.6196	0.5841	0.4949	0.4263	0.3708	0.3113	0.2740
4		0.7823	0.6500	0.6187	0.5393	0.4772	0.4262	0.3704	0.3347
5		0.7892	0.6702	0.6418	0.5695	0.5125	0.4653	0.4130	0.3791
10		0.8029	0.7177	0.6971	0.6440	0.6014	0.5654	0.5247	0.4978
1	4	0.7691	0.5506	0.5013	0.3822	0.2966	0.2323	0.1694	0.1339
2		0.8176	0.6571	0.6195	0.5249	0.4521	0.3933	0.3302	0.2906
3		0.8338	0.7011	0.6696	0.5894	0.5264	0.4744	0.4171	0.3802
4		0.8419	0.7262	0.6985	0.6277	0.5715	0.5246	0.4724	0.4383
5		0.8467	0.7429	0.7179	0.6537	0.6025	0.5596	0.5113	0.4796
10		0.8564	0.7823	0.7643	0.7176	0.6799	0.6479	0.6114	0.5871
1	5	0.8194	0.6201	0.5742	0.4612	0.3770	0.3113	0.2438	0.2035
2		0.8570	0.7121	0.6778	0.5907	0.5227	0.4669	0.4058	0.3667
3		0.8695	0.7500	0.7214	0.6483	0.5903	0.5418	0.4879	0.4527
4		0.8757	0.7718	0.7467	0.6824	0.6309	0.5877	0.5391	0.5071
5		0.8795	0.7862	0.7636	0.7055	0.6588	0.6194	0.5748	0.5453
10		0.8870	0.8205	0.8043	0.7622	0.7281	0.6990	0.6658	0.6435
1	10	0.9134	0.7680	0.7335	0.6456	0.5766	0.5197	0.4569	0.4165
2		0.9311	0.8269	0.8018	0.7371	0.6852	0.6415	0.5922	0.5597
3		0.9370	0.8515	0.8307	0.7770	0.7337	0.6970	0.6552	0.6274
4		0.9399	0.8656	0.8476	0.8007	0.7627	0.7303	0.6934	0.6687
5		0.9416	0.8751	0.8589	0.8167	0.7825	0.7532	0.7197	0.6973
10		0.9452	0.8979	0.8863	0.8561	0.8314	0.8102	0.7858	0.7694
1	25	0.9662	0.8726	0.8500	0.7914	0.7443	0.7043	0.6590	0.6288
2		0.9730	0.9064	0.8902	0.8479	0.8136	0.7843	0.7507	0.7282
3		0.9753	0.9208	0.9074	0.8726	0.8443	0.8200	0.7921	0.7732
4		0.9764	0.9291	0.9175	0.8873	0.8626	0.8413	0.8169	0.8004
5		0.9771	0.9347	0.9244	0.8972	0.8750	0.8559	0.8339	0.8190
10		0.9784	0.9484	0.9410	0.9217	0.9058	0.8922	0.8763	0.8656

Appendix 1-7. Extreme Spread, One Shot Group

Table A1-7.1. Percentiles of the Extreme Spread (*ES*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
2	5.2565	4.6036	4.2919	3.8413	3.4616	3.0349	2.5373	2.3548
3	5.66	5.05	4.74	4.329	3.986	3.599	3.147	2.981
4	5.90	5.30	5.01	4.625	4.291	3.927	3.497	3.342
5	6.04	5.49	5.22	4.83	4.515	4.156	3.743	3.589
6	6.17	5.628	5.36	4.987	4.679	4.332	3.935	3.786
7	6.26	5.73	5.480	5.111	4.811	4.472	4.083	3.938
8	6.34	5.82	5.569	5.222	4.924	4.595	4.211	4.072
9	6.47	5.91	5.665	5.314	5.015	4.692	4.319	4.180
10	6.49	6.00	5.75	5.403	5.106	4.789	4.419	4.285
11	6.56	6.06	5.81	5.462	5.176	4.862	4.499	4.366
12	6.60	6.11	5.878	5.532	5.250	4.935	4.573	4.441
13	6.67	6.16	5.92	5.591	5.312	4.999	4.640	4.510
14	6.71	6.22	5.986	5.648	5.360	5.055	4.701	4.574
15	6.76	6.27	6.037	5.695	5.424	5.115	4.763	4.636
16	6.81	6.30	6.07	5.743	5.474	5.167	4.819	4.690
17	6.86	6.35	6.110	5.774	5.505	5.204	4.861	4.736
18	6.83	6.38	6.15	5.824	5.554	5.252	4.910	4.785
19	6.93	6.42	6.199	5.862	5.592	5.293	4.952	4.829
20	6.96	6.453	6.226	5.897	5.623	5.330	4.994	4.872
21	6.99	6.49	6.26	5.936	5.658	5.365	5.032	4.910
22	7.00	6.50	6.270	5.957	5.696	5.401	5.066	4.946
23	7.01	6.54	6.31	5.994	5.720	5.433	5.101	4.979
24	7.07	6.56	6.34	6.019	5.757	5.466	5.134	5.014
25	7.09	6.60	6.371	6.051	5.784	5.496	5.162	5.044
26	7.10	6.61	6.384	6.067	5.809	5.521	5.193	5.075
27	7.09	6.62	6.401	6.09	5.831	5.553	5.224	5.105
28	7.12	6.66	6.43	6.120	5.859	5.569	5.247	5.129
29	7.16	6.69	6.46	6.14	5.884	5.598	5.276	5.159
30	7.18	6.71	6.49	6.167	5.908	5.624	5.300	5.184

Table A1-7.1. Percentiles of the Extreme Spread (*ES*). (CONTINUED)

n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2	1.6651	1.0727	0.9448	0.6492	0.4530	0.3182	0.2005	0.1416
3	2.341	1.764	1.631	1.306	1.067	0.881	0.691	0.573
4	2.735	2.186	2.057	1.739	1.497	1.302	1.090	0.958
5	3.005	2.474	2.352	2.043	1.804	1.604	1.390	1.255
6	3.222	2.708	2.589	2.289	2.048	1.855	1.634	1.500
7	3.393	2.894	2.776	2.478	2.246	2.057	1.847	1.716
8	3.532	3.045	2.930	2.641	2.411	2.227	2.012	1.878
9	3.657	3.177	3.064	2.779	2.556	2.372	2.164	2.030
10	3.766	3.291	3.181	2.903	2.686	2.502	2.300	2.173
11	3.855	3.392	3.285	3.009	2.791	2.611	2.415	2.286
12	3.942	3.485	3.377	3.103	2.893	2.717	2.520	2.391
13	4.014	3.566	3.459	3.192	2.983	2.805	2.615	2.49
14	4.085	3.636	3.532	3.268	3.062	2.888	2.693	2.569
15	4.150	3.707	3.606	3.345	3.133	2.959	2.768	2.645
16	4.210	3.771	3.668	3.407	3.204	3.037	2.850	2.726
17	4.261	3.827	3.725	3.468	3.266	3.097	2.907	2.784
18	4.314	3.883	3.784	3.527	3.327	3.158	2.969	2.844
19	4.359	3.935	3.835	3.581	3.383	3.220	3.035	2.907
20	4.405	3.981	3.883	3.632	3.438	3.274	3.088	2.971
21	4.449	4.027	3.930	3.683	3.488	3.328	3.142	3.019
22	4.486	4.070	3.973	3.726	3.533	3.374	3.198	3.078
23	4.525	4.111	4.013	3.768	3.578	3.418	3.246	3.126
24	4.561	4.146	4.051	3.808	3.615	3.459	3.283	3.170
25	4.592	4.184	4.089	3.848	3.657	3.500	3.323	3.212
26	4.625	4.219	4.123	3.886	3.697	3.540	3.369	3.257
27	4.657	4.252	4.157	3.920	3.732	3.576	3.407	3.301
28	4.688	4.287	4.192	3.954	3.767	3.613	3.442	3.329
29	4.715	4.316	4.222	3.988	3.802	3.648	3.48	3.36
30	4.744	4.347	4.253	4.017	3.837	3.683	3.515	3.405

Table A1-7.2. Monte Carlo Errors of the Percentiles of the Extreme Spread (*ES*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2																
3	0.09	0.05	0.04	0.02	0.02	0.014	0.012	0.011	0.010	0.008	0.007	0.009	0.006	0.010	0.014	0.018
4	0.09	0.06	0.03	0.014	0.010	0.007	0.007	0.006	0.007	0.008	0.006	0.007	0.008	0.009	0.009	0.013
5	0.06	0.04	0.03	0.03	0.02	0.011	0.009	0.006	0.008	0.007	0.006	0.008	0.012	0.010	0.02	0.02
6	0.07	0.02	0.03	0.015	0.007	0.007	0.009	0.009	0.005	0.007	0.007	0.006	0.010	0.011	0.017	0.02
7	0.06	0.04	0.02	0.02	0.008	0.010	0.011	0.008	0.007	0.007	0.008	0.008	0.005	0.009	0.013	0.014
8	0.07	0.04	0.02	0.013	0.008	0.005	0.006	0.006	0.006	0.008	0.009	0.009	0.009	0.011	0.014	0.014
9	0.04	0.03	0.013	0.017	0.013	0.008	0.011	0.009	0.004	0.005	0.005	0.004	0.008	0.011	0.015	0.02
10	0.08	0.03	0.03	0.018	0.012	0.010	0.009	0.008	0.006	0.008	0.007	0.008	0.010	0.014	0.015	0.013
11	0.06	0.04	0.03	0.016	0.014	0.011	0.009	0.009	0.009	0.007	0.007	0.010	0.009	0.009	0.017	0.02
12	0.06	0.03	0.018	0.02	0.015	0.007	0.006	0.007	0.006	0.008	0.008	0.009	0.011	0.013	0.018	0.02
13	0.05	0.04	0.03	0.02	0.014	0.012	0.008	0.007	0.006	0.006	0.007	0.009	0.012	0.009	0.013	0.03
14	0.05	0.03	0.02	0.014	0.01	0.012	0.007	0.007	0.006	0.005	0.006	0.006	0.006	0.012	0.01	0.015
15	0.08	0.03	0.019	0.02	0.014	0.008	0.006	0.006	0.005	0.007	0.008	0.009	0.007	0.009	0.014	0.016
16	0.04	0.03	0.03	0.02	0.016	0.007	0.004	0.006	0.006	0.004	0.005	0.007	0.008	0.009	0.009	0.012
17	0.06	0.04	0.02	0.015	0.006	0.010	0.009	0.008	0.006	0.006	0.007	0.010	0.011	0.012	0.011	0.019
18	0.08	0.04	0.03	0.018	0.016	0.012	0.008	0.008	0.007	0.007	0.007	0.007	0.007	0.011	0.016	0.017
19	0.07	0.03	0.02	0.019	0.013	0.007	0.008	0.007	0.004	0.007	0.006	0.009	0.008	0.011	0.011	0.02
20	0.08	0.02	0.018	0.017	0.014	0.010	0.008	0.006	0.005	0.006	0.004	0.005	0.010	0.011	0.011	0.015
21	0.08	0.04	0.03	0.016	0.013	0.014	0.006	0.007	0.006	0.009	0.009	0.009	0.009	0.012	0.005	0.014
22	0.06	0.03	0.02	0.02	0.011	0.008	0.003	0.005	0.005	0.006	0.006	0.008	0.008	0.012	0.013	0.012
23	0.08	0.04	0.03	0.019	0.014	0.008	0.006	0.008	0.006	0.005	0.006	0.007	0.009	0.013	0.012	0.019
24	0.08	0.04	0.03	0.015	0.015	0.009	0.006	0.006	0.008	0.006	0.005	0.007	0.008	0.011	0.013	0.006
25	0.07	0.04	0.017	0.02	0.014	0.006	0.005	0.006	0.007	0.005	0.005	0.008	0.012	0.013	0.015	0.018
26	0.04	0.03	0.02	0.016	0.012	0.012	0.006	0.006	0.005	0.004	0.005	0.005	0.006	0.009	0.013	0.011
27	0.06	0.03	0.02	0.03	0.02	0.012	0.008	0.007	0.005	0.006	0.007	0.011	0.011	0.014	0.017	0.02
28	0.05	0.03	0.03	0.019	0.015	0.013	0.010	0.007	0.007	0.007	0.008	0.004	0.006	0.009	0.012	0.011
29	0.04	0.04	0.03	0.03	0.019	0.012	0.007	0.005	0.007	0.003	0.003	0.006	0.007	0.005	0.03	0.050
30	0.05	0.03	0.03	0.013	0.011	0.007	0.003	0.003	0.006	0.006	0.006	0.005	0.007	0.009	0.009	0.02

Appendix 1-8. Average Extreme Spread, Multiple Shot Groups

Table A1-8.1.1. Percentiles of the Average Extreme Spread (AES).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
1	2	5.2565	4.6036	4.2919	3.8413	3.4616	3.0349	2.5373	2.3548
2		4.17	3.71	3.500	3.192	2.934	2.650	2.313	2.192
3		3.630	3.31	3.156	2.916	2.711	2.480	2.214	2.117
4		3.38	3.090	2.952	2.750	2.579	2.382	2.155	2.071
5		3.200	2.933	2.811	2.640	2.487	2.315	2.116	2.041
10		2.748	2.571	2.490	2.370	2.269	2.152	2.016	1.965
1	3	5.66	5.05	4.74	4.329	3.986	3.599	3.147	2.981
2		4.60	4.21	4.014	3.739	3.502	3.241	2.931	2.814
3		4.18	3.85	3.698	3.481	3.291	3.081	2.835	2.745
4		3.91	3.643	3.519	3.331	3.173	2.991	2.781	2.703
5		3.75	3.513	3.396	3.232	3.092	2.930	2.742	2.671
10		3.343	3.172	3.092	2.979	2.884	2.773	2.644	2.595
1	4	5.90	5.30	5.01	4.625	4.291	3.927	3.497	3.342
2		4.90	4.50	4.32	4.057	3.837	3.588	3.295	3.187
3		4.48	4.15	4.017	3.813	3.638	3.439	3.206	3.118
4		4.221	3.972	3.853	3.673	3.520	3.353	3.149	3.075
5		4.072	3.838	3.731	3.574	3.443	3.291	3.114	3.045
10		3.678	3.523	3.451	3.342	3.250	3.147	3.021	2.974
1	5	6.04	5.49	5.22	4.83	4.515	4.156	3.743	3.589
2		5.10	4.74	4.553	4.295	4.077	3.836	3.550	3.445
3		4.67	4.38	4.25	4.048	3.881	3.688	3.462	3.377
4		4.45	4.202	4.080	3.913	3.765	3.601	3.408	3.335
5		4.30	4.075	3.972	3.821	3.691	3.548	3.374	3.311
10		3.918	3.772	3.699	3.594	3.505	3.405	3.285	3.240
1	10	6.49	6.00	5.75	5.403	5.106	4.789	4.419	4.285
2		5.65	5.298	5.140	4.905	4.710	4.495	4.243	4.151
3		5.28	5.010	4.880	4.695	4.542	4.369	4.168	4.091
4		5.063	4.838	4.730	4.572	4.442	4.294	4.120	4.055
5		4.92	4.719	4.629	4.488	4.373	4.243	4.088	4.029
10		4.574	4.438	4.376	4.283	4.204	4.113	4.006	3.965
1	25	7.09	6.60	6.371	6.051	5.784	5.496	5.162	5.044
2		6.27	5.958	5.813	5.602	5.430	5.242	5.016	4.935
3		5.931	5.690	5.581	5.419	5.283	5.130	4.949	4.883
4		5.754	5.544	5.443	5.306	5.192	5.061	4.909	4.852
5		5.618	5.442	5.358	5.237	5.133	5.019	4.883	4.831
10		5.321	5.196	5.140	5.058	4.987	4.907	4.812	4.777

Table A1-8.1. Percentiles of the Average Extreme Spread (AES). (CONTINUED)

k	n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2	1.6651	1.0727	0.9448	0.6492	0.4530	0.3182	0.2005	0.1416
2		1.723	1.301	1.204	0.966	0.788	0.649	0.505	0.418
3		1.740	1.392	1.310	1.106	0.952	0.826	0.692	0.610
4		1.748	1.445	1.373	1.196	1.056	0.942	0.817	0.739
5		1.752	1.482	1.417	1.256	1.126	1.021	0.904	0.831
10		1.762	1.570	1.524	1.404	1.307	1.226	1.135	1.077
1	3	2.341	1.764	1.631	1.306	1.067	0.881	0.691	0.573
2		2.373	1.962	1.865	1.621	1.428	1.274	1.106	0.999
3		2.386	2.048	1.968	1.762	1.602	1.467	1.322	1.223
4		2.393	2.097	2.027	1.848	1.706	1.587	1.457	1.367
5		2.395	2.132	2.067	1.904	1.775	1.667	1.548	1.468
10		2.401	2.215	2.169	2.051	1.957	1.876	1.785	1.725
1	4	2.735	2.186	2.057	1.739	1.497	1.302	1.090	0.958
2		2.766	2.368	2.274	2.036	1.850	1.695	1.526	1.410
3		2.775	2.450	2.372	2.172	2.016	1.882	1.727	1.632
4		2.780	2.497	2.429	2.253	2.113	1.997	1.858	1.768
5		2.782	2.530	2.469	2.310	2.183	2.077	1.954	1.875
10		2.787	2.607	2.563	2.450	2.359	2.280	2.189	2.128
1	5	3.005	2.474	2.352	2.043	1.804	1.604	1.390	1.255
2		3.040	2.657	2.566	2.336	2.154	2.003	1.832	1.716
3		3.048	2.732	2.658	2.465	2.311	2.183	2.042	1.942
4		3.049	2.778	2.712	2.543	2.409	2.293	2.165	2.076
5		3.056	2.811	2.751	2.599	2.477	2.373	2.255	2.173
10		3.061	2.886	2.844	2.733	2.644	2.569	2.478	2.419
1	10	3.766	3.291	3.181	2.903	2.686	2.502	2.300	2.173
2		3.789	3.450	3.368	3.160	2.994	2.855	2.695	2.594
3		3.796	3.515	3.447	3.275	3.136	3.017	2.884	2.796
4		3.800	3.556	3.498	3.346	3.224	3.120	3.001	2.927
5		3.802	3.582	3.530	3.394	3.285	3.192	3.085	3.016
10		3.806	3.650	3.612	3.514	3.434	3.365	3.287	3.234
1	25	4.592	4.184	4.089	3.848	3.657	3.500	3.323	3.212
2		4.618	4.321	4.252	4.072	3.927	3.808	3.672	3.582
3		4.625	4.380	4.322	4.172	4.051	3.950	3.838	3.761
4		4.629	4.416	4.365	4.234	4.130	4.041	3.940	3.872
5		4.631	4.440	4.394	4.275	4.179	4.097	4.003	3.945
10		4.636	4.501	4.467	4.381	4.311	4.251	4.184	4.138

Table A1-8.2. Monte Carlo Errors of Percentiles of the Average Extreme Spread (AES).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2																
2	2	0.06	0.03	0.02	0.007	0.008	0.008	0.005	0.005	0.005	0.004	0.006	0.006	0.009	0.008	0.008	0.009
3	2	0.02	0.03	0.018	0.019	0.010	0.009	0.005	0.005	0.004	0.004	0.004	0.004	0.007	0.006	0.010	0.016
4	2	0.04	0.013	0.010	0.012	0.009	0.007	0.005	0.004	0.005	0.004	0.005	0.004	0.005	0.004	0.006	0.009
5	2	0.017	0.02	0.016	0.012	0.008	0.006	0.005	0.004	0.004	0.003	0.004	0.004	0.003	0.004	0.008	0.012
10	2	0.019	0.009	0.010	0.005	0.005	0.003	0.003	0.003	0.003	0.0017	0.0017	0.0018	0.003	0.004	0.005	0.006
1	3	0.09	0.05	0.04	0.02	0.02	0.014	0.012	0.011	0.010	0.008	0.007	0.009	0.006	0.010	0.014	0.018
2	3	0.06	0.03	0.02	0.009	0.012	0.009	0.009	0.010	0.007	0.007	0.006	0.008	0.011	0.015	0.013	0.013
3	3	0.06	0.03	0.016	0.015	0.010	0.007	0.006	0.004	0.004	0.004	0.005	0.004	0.007	0.011	0.014	0.014
4	3	0.03	0.019	0.012	0.011	0.007	0.007	0.004	0.004	0.005	0.006	0.006	0.006	0.004	0.008	0.007	0.013
5	3	0.04	0.016	0.013	0.007	0.008	0.004	0.005	0.005	0.005	0.003	0.002	0.002	0.002	0.006	0.007	0.009
10	3	0.018	0.015	0.009	0.005	0.004	0.003	0.003	0.003	0.003	0.003	0.004	0.003	0.004	0.004	0.007	0.007
1	4	0.09	0.06	0.03	0.014	0.010	0.007	0.007	0.006	0.007	0.008	0.006	0.007	0.008	0.009	0.009	0.013
2	4	0.05	0.03	0.03	0.02	0.013	0.009	0.009	0.007	0.004	0.003	0.005	0.004	0.007	0.011	0.012	0.017
3	4	0.03	0.03	0.02	0.013	0.011	0.009	0.006	0.004	0.005	0.005	0.004	0.007	0.007	0.007	0.010	0.009
4	4	0.019	0.015	0.011	0.008	0.007	0.005	0.005	0.005	0.0019	0.003	0.003	0.005	0.005	0.009	0.012	0.010
5	4	0.02	0.018	0.012	0.007	0.007	0.003	0.002	0.003	0.003	0.004	0.003	0.004	0.004	0.006	0.009	0.012
10	4	0.013	0.009	0.007	0.007	0.006	0.003	0.003	0.002	0.0015	0.0017	0.0019	0.004	0.004	0.005	0.006	0.009
1	5	0.06	0.04	0.03	0.03	0.02	0.011	0.009	0.006	0.008	0.007	0.006	0.008	0.012	0.010	0.02	0.02
2	5	0.05	0.03	0.017	0.013	0.013	0.006	0.004	0.003	0.003	0.004	0.004	0.006	0.005	0.003	0.009	0.010
3	5	0.03	0.03	0.03	0.014	0.009	0.006	0.004	0.004	0.003	0.004	0.004	0.004	0.005	0.007	0.012	0.010
4	5	0.03	0.010	0.011	0.010	0.007	0.004	0.004	0.003	0.003	0.002	0.004	0.005	0.006	0.007	0.010	0.014
5	5	0.03	0.02	0.011	0.007	0.002	0.003	0.004	0.004	0.003	0.003	0.004	0.004	0.004	0.004	0.003	0.005
10	5	0.014	0.010	0.009	0.005	0.003	0.004	0.003	0.003	0.002	0.003	0.002	0.0019	0.003	0.007	0.008	0.009
1	10	0.08	0.03	0.03	0.018	0.012	0.010	0.009	0.008	0.006	0.008	0.007	0.008	0.010	0.014	0.015	0.013
2	10	0.06	0.015	0.013	0.015	0.012	0.010	0.005	0.004	0.004	0.005	0.006	0.004	0.005	0.008	0.011	0.013
3	10	0.05	0.017	0.018	0.011	0.007	0.004	0.004	0.005	0.004	0.004	0.004	0.004	0.005	0.007	0.004	0.010
4	10	0.02	0.010	0.010	0.008	0.005	0.004	0.004	0.004	0.003	0.002	0.004	0.004	0.005	0.006	0.010	0.012
5	10	0.03	0.016	0.016	0.008	0.007	0.006	0.004	0.003	0.005	0.0018	0.0012	0.0018	0.003	0.004	0.007	0.011
10	10	0.015	0.013	0.010	0.005	0.004	0.0017	0.0016	0.0015	0.002	0.0014	0.002	0.002	0.004	0.004	0.005	0.008
1	25	0.07	0.04	0.017	0.02	0.014	0.006	0.005	0.006	0.007	0.005	0.005	0.008	0.012	0.013	0.015	0.018
2	25	0.03	0.015	0.014	0.011	0.006	0.004	0.005	0.006	0.004	0.004	0.004	0.004	0.007	0.007	0.011	0.016
3	25	0.019	0.012	0.011	0.005	0.004	0.004	0.0016	0.002	0.004	0.003	0.005	0.005	0.005	0.005	0.007	0.013
4	25	0.011	0.013	0.009	0.005	0.005	0.005	0.005	0.004	0.005	0.003	0.003	0.004	0.005	0.005	0.006	0.009
5	25	0.019	0.011	0.009	0.007	0.004	0.004	0.002	0.003	0.003	0.003	0.002	0.003	0.003	0.003	0.007	0.013
10	25	0.014	0.008	0.006	0.004	0.003	0.002	0.0013	0.0016	0.002	0.003	0.002	0.004	0.004	0.004	0.004	0.005

Appendix 1-9. Mean Radius, One Shot Group

Table A1-9.1. Percentiles of the Mean Radius (*MR*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
2	2.6283	2.3018	2.1460	1.9206	1.7308	1.5174	1.2686	1.1774
3	2.35	2.106	1.990	1.824	1.683	1.522	1.335	1.266
4	2.23	2.013	1.917	1.772	1.651	1.516	1.354	1.296
5	2.117	1.944	1.855	1.728	1.623	1.501	1.360	1.308
6	2.044	1.885	1.808	1.696	1.602	1.493	1.364	1.317
7	1.991	1.847	1.776	1.672	1.583	1.483	1.364	1.320
8	1.956	1.815	1.744	1.645	1.565	1.474	1.364	1.324
9	1.909	1.783	1.720	1.631	1.552	1.466	1.363	1.324
10	1.881	1.755	1.697	1.612	1.541	1.458	1.360	1.324
11	1.857	1.739	1.683	1.600	1.531	1.452	1.359	1.324
12	1.821	1.720	1.665	1.586	1.519	1.445	1.356	1.324
13	1.805	1.695	1.647	1.573	1.511	1.440	1.354	1.323
14	1.785	1.684	1.634	1.565	1.505	1.435	1.354	1.323
15	1.762	1.670	1.623	1.553	1.496	1.429	1.351	1.322
16	1.755	1.657	1.612	1.546	1.489	1.425	1.349	1.321
17	1.729	1.643	1.602	1.537	1.484	1.422	1.349	1.321
18	1.720	1.633	1.591	1.529	1.477	1.417	1.347	1.320
19	1.706	1.624	1.584	1.524	1.472	1.414	1.345	1.319
20	1.702	1.615	1.574	1.518	1.468	1.412	1.344	1.318
21	1.686	1.610	1.570	1.512	1.464	1.407	1.342	1.318
22	1.683	1.604	1.565	1.509	1.461	1.406	1.342	1.317
23	1.673	1.594	1.556	1.502	1.455	1.403	1.340	1.317
24	1.660	1.585	1.549	1.496	1.451	1.400	1.339	1.316
25	1.655	1.581	1.545	1.494	1.449	1.399	1.339	1.316
26	1.651	1.575	1.540	1.489	1.444	1.395	1.336	1.314
27	1.634	1.567	1.532	1.484	1.442	1.394	1.336	1.314
28	1.628	1.563	1.528	1.480	1.438	1.391	1.335	1.314
29	1.630	1.563	1.528	1.477	1.436	1.388	1.333	1.312
30	1.625	1.552	1.519	1.472	1.432	1.386	1.332	1.311

Table A1-9.1. Percentiles of the Mean Radius (*MR*). (CONTINUED)

n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2	0.8326	0.5364	0.4724	0.3246	0.2265	0.1591	0.1003	0.0708
3	0.997	0.754	0.697	0.561	0.459	0.381	0.298	0.247
4	1.066	0.854	0.804	0.680	0.586	0.511	0.428	0.374
5	1.107	0.916	0.872	0.759	0.669	0.599	0.518	0.467
6	1.131	0.958	0.917	0.811	0.729	0.662	0.589	0.543
7	1.149	0.988	0.949	0.850	0.772	0.706	0.633	0.586
8	1.163	1.011	0.975	0.883	0.809	0.746	0.680	0.634
9	1.173	1.030	0.996	0.907	0.839	0.780	0.716	0.674
10	1.181	1.046	1.013	0.929	0.862	0.806	0.744	0.702
11	1.188	1.058	1.027	0.946	0.882	0.827	0.766	0.726
12	1.193	1.068	1.038	0.962	0.900	0.849	0.792	0.752
13	1.198	1.079	1.050	0.975	0.916	0.867	0.809	0.769
14	1.203	1.088	1.060	0.988	0.930	0.882	0.827	0.791
15	1.205	1.094	1.067	0.997	0.941	0.894	0.841	0.806
16	1.209	1.101	1.075	1.008	0.954	0.908	0.856	0.823
17	1.212	1.107	1.082	1.016	0.962	0.916	0.868	0.834
18	1.214	1.111	1.087	1.023	0.972	0.928	0.879	0.845
19	1.217	1.117	1.093	1.030	0.980	0.937	0.889	0.855
20	1.218	1.121	1.097	1.036	0.986	0.945	0.897	0.867
21	1.220	1.125	1.101	1.042	0.994	0.954	0.910	0.878
22	1.221	1.129	1.107	1.048	1.002	0.962	0.914	0.883
23	1.223	1.132	1.110	1.053	1.007	0.967	0.921	0.891
24	1.224	1.136	1.114	1.058	1.011	0.972	0.929	0.899
25	1.225	1.139	1.117	1.062	1.018	0.979	0.936	0.906
26	1.226	1.141	1.120	1.067	1.023	0.987	0.944	0.915
27	1.227	1.143	1.123	1.070	1.028	0.992	0.949	0.921
28	1.229	1.146	1.126	1.074	1.032	0.995	0.956	0.928
29	1.229	1.148	1.128	1.077	1.036	1.000	0.959	0.932
30	1.230	1.150	1.131	1.080	1.039	1.004	0.965	0.939

Table A1-9.2. Monte Carlo Errors of Percentiles of the Mean Radius (*MR*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2																
3	0.03	0.009	0.006	0.008	0.005	0.004	0.003	0.004	0.003	0.003	0.003	0.003	0.003	0.004	0.006	0.006
4	0.03	0.018	0.014	0.009	0.004	0.004	0.003	0.002	0.003	0.003	0.003	0.003	0.004	0.007	0.007	0.009
5	0.02	0.015	0.006	0.007	0.006	0.006	0.004	0.005	0.003	0.003	0.003	0.003	0.005	0.005	0.005	0.007
6	0.02	0.011	0.005	0.004	0.002	0.002	0.0017	0.0018	0.002	0.002	0.003	0.002	0.0016	0.003	0.004	0.005
7	0.018	0.009	0.005	0.003	0.003	0.002	0.002	0.0018	0.003	0.002	0.003	0.003	0.0017	0.003	0.006	0.007
8	0.019	0.011	0.008	0.005	0.005	0.004	0.003	0.002	0.0015	0.0013	0.0013	0.002	0.003	0.003	0.005	0.008
9	0.019	0.009	0.010	0.005	0.003	0.003	0.002	0.002	0.0009	0.002	0.002	0.0018	0.003	0.003	0.005	0.005
10	0.018	0.007	0.007	0.005	0.004	0.002	0.002	0.002	0.002	0.002	0.0019	0.0012	0.0015	0.003	0.005	0.002
11	0.008	0.009	0.007	0.006	0.004	0.002	0.002	0.002	0.0019	0.0017	0.0018	0.0011	0.002	0.004	0.004	0.004
12	0.015	0.007	0.006	0.004	0.003	0.003	0.002	0.0019	0.002	0.0016	0.0015	0.002	0.002	0.003	0.003	0.005
13	0.011	0.005	0.006	0.003	0.002	0.0016	0.0014	0.0015	0.0015	0.0016	0.0015	0.002	0.003	0.002	0.003	0.005
14	0.011	0.008	0.007	0.004	0.002	0.002	0.002	0.003	0.0019	0.0015	0.0018	0.0019	0.002	0.002	0.003	0.004
15	0.014	0.011	0.009	0.005	0.003	0.002	0.0019	0.0018	0.0011	0.0014	0.0019	0.0017	0.003	0.003	0.004	0.005
16	0.017	0.007	0.006	0.006	0.005	0.003	0.002	0.002	0.0015	0.0006	0.0008	0.0012	0.0017	0.003	0.004	0.005
17	0.009	0.006	0.004	0.002	0.002	0.0014	0.0016	0.002	0.0019	0.0015	0.0015	0.0017	0.002	0.004	0.005	0.007
18	0.014	0.005	0.005	0.002	0.002	0.002	0.002	0.0019	0.0008	0.0009	0.0009	0.0015	0.0012	0.002	0.004	0.004
19	0.012	0.008	0.005	0.003	0.003	0.002	0.0018	0.002	0.0015	0.0015	0.0015	0.0016	0.002	0.002	0.004	0.003
20	0.011	0.005	0.004	0.002	0.0015	0.0015	0.0009	0.0011	0.0016	0.0015	0.0012	0.0013	0.0018	0.003	0.004	0.004
21	0.005	0.006	0.003	0.004	0.003	0.002	0.0017	0.0014	0.0012	0.0015	0.0009	0.0011	0.0019	0.0016	0.003	0.004
22	0.016	0.008	0.006	0.003	0.003	0.002	0.0019	0.002	0.0013	0.0011	0.0013	0.0015	0.003	0.002	0.002	0.004
23	0.012	0.003	0.004	0.003	0.002	0.0018	0.0010	0.0011	0.0009	0.0008	0.0012	0.0016	0.0016	0.0014	0.002	0.005
24	0.010	0.005	0.003	0.0015	0.0019	0.0016	0.0012	0.0012	0.0009	0.0011	0.0011	0.0017	0.0013	0.003	0.003	0.004
25	0.009	0.004	0.003	0.003	0.004	0.0019	0.0013	0.0009	0.0010	0.0014	0.0013	0.0016	0.0015	0.0012	0.003	0.004
26	0.010	0.005	0.003	0.002	0.0019	0.0012	0.0011	0.0014	0.0011	0.0009	0.0011	0.0009	0.0012	0.0015	0.003	0.003
27	0.009	0.006	0.004	0.003	0.003	0.002	0.002	0.0015	0.0013	0.0008	0.0009	0.0006	0.0014	0.0019	0.004	0.004
28	0.008	0.005	0.002	0.0013	0.0017	0.0013	0.0009	0.0007	0.0008	0.0014	0.0012	0.0011	0.0016	0.002	0.002	0.001
29	0.009	0.005	0.002	0.003	0.002	0.0017	0.0015	0.0014	0.0012	0.0014	0.0014	0.0010	0.0018	0.002	0.003	0.004
30	0.008	0.005	0.004	0.002	0.002	0.0019	0.0011	0.0008	0.0007	0.0008	0.0009	0.0015	0.002	0.002	0.003	0.003

Appendix 1-10. Average Mean Radius, Multiple Shot Groups

Table A1-10.1. Percentiles of the Average Mean Radius (AMR).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
1	2	2.6283	2.3018	2.1460	1.9206	1.7308	1.5174	1.2686	1.1774
2		2.08	1.852	1.743	1.592	1.466	1.326	1.160	1.098
3		1.821	1.651	1.575	1.453	1.351	1.239	1.107	1.058
4		1.685	1.544	1.473	1.371	1.286	1.191	1.079	1.037
5		1.602	1.467	1.404	1.316	1.241	1.159	1.059	1.021
10		1.375	1.291	1.248	1.186	1.135	1.077	1.008	0.982
1	3	2.35	2.106	1.990	1.824	1.683	1.522	1.335	1.266
2		1.943	1.766	1.688	1.576	1.479	1.370	1.242	1.195
3		1.748	1.624	1.562	1.472	1.394	1.307	1.204	1.165
4		1.651	1.540	1.486	1.408	1.343	1.267	1.179	1.146
5		1.582	1.484	1.436	1.366	1.307	1.242	1.163	1.134
10		1.407	1.341	1.308	1.262	1.222	1.177	1.123	1.102
1	4	2.23	2.013	1.917	1.772	1.651	1.516	1.354	1.296
2		1.869	1.728	1.660	1.563	1.480	1.387	1.277	1.236
3		1.714	1.599	1.547	1.469	1.402	1.328	1.241	1.208
4		1.625	1.532	1.484	1.419	1.362	1.297	1.221	1.192
5		1.566	1.480	1.441	1.381	1.332	1.274	1.206	1.180
10		1.417	1.359	1.331	1.292	1.257	1.218	1.171	1.154
1	5	2.117	1.944	1.855	1.728	1.623	1.501	1.360	1.308
2		1.813	1.694	1.630	1.543	1.472	1.390	1.292	1.256
3		1.677	1.578	1.530	1.463	1.404	1.339	1.262	1.232
4		1.588	1.514	1.476	1.415	1.365	1.309	1.242	1.217
5		1.550	1.473	1.436	1.385	1.340	1.290	1.230	1.207
10		1.412	1.365	1.342	1.305	1.275	1.240	1.198	1.182
1	10	1.881	1.755	1.697	1.612	1.541	1.458	1.360	1.324
2		1.662	1.583	1.544	1.484	1.435	1.378	1.311	1.286
3		1.576	1.510	1.476	1.429	1.388	1.343	1.288	1.268
4		1.521	1.465	1.436	1.396	1.362	1.322	1.275	1.258
5		1.483	1.433	1.408	1.373	1.343	1.308	1.266	1.251
10		1.396	1.361	1.344	1.319	1.297	1.273	1.244	1.233
1	25	1.655	1.581	1.545	1.494	1.449	1.399	1.339	1.316
2		1.527	1.476	1.450	1.414	1.382	1.348	1.305	1.289
3		1.468	1.427	1.408	1.379	1.354	1.325	1.291	1.278
4		1.437	1.401	1.383	1.358	1.336	1.312	1.283	1.272
5		1.413	1.382	1.366	1.344	1.325	1.303	1.277	1.267
10		1.358	1.335	1.325	1.309	1.296	1.281	1.263	1.256

Table A1-10.1. Percentiles of the Average Mean Radius (AMR). (CONTINUED)

k	n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2	0.8326	0.5364	0.4724	0.3246	0.2265	0.1591	0.1003	0.0708
2		0.862	0.650	0.601	0.481	0.393	0.325	0.255	0.215
3		0.869	0.696	0.655	0.553	0.476	0.414	0.347	0.308
4		0.875	0.722	0.687	0.596	0.526	0.470	0.407	0.368
5		0.876	0.739	0.707	0.626	0.561	0.508	0.450	0.412
10		0.881	0.784	0.762	0.702	0.654	0.615	0.568	0.537
1	3	0.997	0.754	0.697	0.561	0.459	0.381	0.298	0.247
2		1.011	0.837	0.796	0.692	0.611	0.544	0.473	0.428
3		1.014	0.872	0.839	0.752	0.684	0.629	0.564	0.525
4		1.016	0.893	0.864	0.788	0.728	0.678	0.620	0.582
5		1.018	0.908	0.881	0.812	0.757	0.710	0.659	0.624
10		1.021	0.942	0.923	0.874	0.833	0.799	0.762	0.736
1	4	1.066	0.854	0.804	0.680	0.586	0.511	0.428	0.374
2		1.077	0.925	0.889	0.798	0.725	0.666	0.598	0.555
3		1.079	0.956	0.926	0.849	0.788	0.739	0.681	0.643
4		1.081	0.973	0.946	0.880	0.827	0.782	0.731	0.695
5		1.081	0.986	0.962	0.902	0.852	0.810	0.763	0.733
10		1.084	1.016	0.999	0.955	0.921	0.891	0.856	0.833
1	5	1.107	0.916	0.872	0.759	0.669	0.599	0.518	0.467
2		1.114	0.978	0.945	0.862	0.797	0.742	0.678	0.640
3		1.116	1.006	0.979	0.910	0.855	0.809	0.756	0.721
4		1.118	1.021	0.998	0.937	0.889	0.849	0.802	0.771
5		1.118	1.032	1.011	0.956	0.912	0.875	0.834	0.806
10		1.119	1.058	1.043	1.004	0.973	0.946	0.915	0.894
1	10	1.181	1.046	1.013	0.929	0.862	0.806	0.744	0.702
2		1.185	1.089	1.065	1.004	0.956	0.914	0.867	0.835
3		1.187	1.108	1.088	1.038	0.997	0.963	0.923	0.897
4		1.187	1.119	1.102	1.058	1.023	0.991	0.956	0.933
5		1.188	1.126	1.111	1.072	1.040	1.012	0.981	0.959
10		1.188	1.145	1.134	1.106	1.083	1.064	1.041	1.026
1	25	1.225	1.139	1.117	1.062	1.018	0.979	0.936	0.906
2		1.226	1.165	1.150	1.111	1.079	1.052	1.020	0.998
3		1.227	1.176	1.164	1.132	1.105	1.083	1.056	1.038
4		1.227	1.184	1.173	1.144	1.121	1.101	1.079	1.063
5		1.227	1.188	1.178	1.153	1.132	1.115	1.094	1.080
10		1.228	1.200	1.193	1.175	1.160	1.148	1.133	1.123

Table A1-10.2. Monte Carlo Errors of the Percentiles of the Average Mean Radius (AMR).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2																
2	2	0.03	0.011	0.011	0.008	0.006	0.006	0.003	0.004	0.004	0.004	0.003	0.003	0.003	0.004	0.002	0.003
3	2	0.02	0.010	0.005	0.007	0.004	0.003	0.002	0.002	0.002	0.003	0.002	0.002	0.002	0.003	0.003	0.007
4	2	0.014	0.012	0.010	0.004	0.003	0.003	0.002	0.003	0.0011	0.0014	0.0016	0.002	0.003	0.003	0.003	0.003
5	2	0.014	0.011	0.007	0.004	0.004	0.003	0.002	0.003	0.0015	0.002	0.0016	0.003	0.003	0.004	0.006	0.005
10	2	0.015	0.008	0.006	0.002	0.0011	0.0012	0.0008	0.0012	0.0011	0.0011	0.0008	0.0011	0.0011	0.0016	0.003	0.004
1	3	0.03	0.009	0.006	0.008	0.005	0.004	0.003	0.004	0.003	0.003	0.003	0.003	0.003	0.004	0.006	0.006
2	3	0.019	0.009	0.007	0.007	0.005	0.003	0.0018	0.003	0.002	0.003	0.002	0.003	0.005	0.006	0.006	0.009
3	3	0.018	0.011	0.010	0.005	0.003	0.003	0.003	0.002	0.0016	0.0010	0.0015	0.0011	0.002	0.004	0.004	0.004
4	3	0.012	0.009	0.006	0.004	0.004	0.004	0.003	0.0019	0.0011	0.0009	0.0014	0.002	0.003	0.002	0.004	0.005
5	3	0.015	0.006	0.005	0.002	0.002	0.0013	0.0017	0.0018	0.0015	0.0019	0.0018	0.003	0.002	0.002	0.0019	0.004
10	3	0.007	0.007	0.003	0.002	0.0016	0.0012	0.0010	0.0008	0.0009	0.0009	0.0011	0.0014	0.0016	0.0016	0.002	0.003
1	4	0.03	0.018	0.014	0.009	0.004	0.004	0.003	0.002	0.003	0.003	0.003	0.003	0.004	0.007	0.007	0.009
2	4	0.010	0.009	0.006	0.005	0.003	0.004	0.003	0.003	0.002	0.0017	0.002	0.0010	0.003	0.004	0.005	0.005
3	4	0.012	0.006	0.004	0.002	0.0017	0.0013	0.0015	0.0016	0.002	0.0010	0.0013	0.0018	0.003	0.003	0.005	0.005
4	4	0.007	0.003	0.004	0.003	0.003	0.002	0.0014	0.0012	0.0014	0.002	0.0018	0.0014	0.0015	0.002	0.003	0.004
5	4	0.012	0.003	0.003	0.003	0.002	0.0019	0.0014	0.0011	0.0013	0.0015	0.0016	0.0016	0.002	0.002	0.003	0.004
10	4	0.007	0.003	0.002	0.0016	0.0019	0.0015	0.0012	0.0009	0.0009	0.0008	0.0009	0.0018	0.002	0.002	0.003	0.005
1	5	0.02	0.015	0.006	0.007	0.006	0.006	0.004	0.005	0.003	0.003	0.003	0.003	0.005	0.005	0.005	0.007
2	5	0.018	0.010	0.007	0.003	0.004	0.004	0.003	0.003	0.003	0.002	0.002	0.003	0.003	0.003	0.004	0.005
3	5	0.015	0.010	0.006	0.005	0.003	0.003	0.0019	0.0018	0.0014	0.0019	0.002	0.003	0.003	0.004	0.007	0.008
4	5	0.011	0.005	0.004	0.003	0.002	0.0017	0.0014	0.0013	0.0010	0.0014	0.0014	0.0018	0.0019	0.003	0.003	0.004
5	5	0.013	0.006	0.003	0.003	0.003	0.0018	0.0017	0.0012	0.0006	0.0011	0.0013	0.0015	0.003	0.002	0.004	0.003
10	5	0.005	0.002	0.0017	0.003	0.0014	0.0010	0.0007	0.0008	0.0011	0.0011	0.0008	0.0012	0.0011	0.0013	0.0018	0.002
1	10	0.018	0.007	0.007	0.005	0.004	0.002	0.002	0.002	0.002	0.002	0.0019	0.0012	0.0015	0.003	0.005	0.002
2	10	0.006	0.004	0.004	0.003	0.003	0.002	0.0017	0.0013	0.0011	0.0014	0.0012	0.0018	0.0010	0.0018	0.003	0.004
3	10	0.010	0.005	0.004	0.004	0.003	0.002	0.0017	0.0013	0.0015	0.0009	0.0009	0.0012	0.0014	0.002	0.003	0.005
4	10	0.009	0.004	0.002	0.003	0.0019	0.0008	0.0011	0.0010	0.0010	0.0009	0.0009	0.0012	0.0012	0.0018	0.0019	0.004
5	10	0.005	0.005	0.003	0.002	0.0015	0.0010	0.0008	0.0006	0.0005	0.0007	0.0008	0.0010	0.0010	0.0014	0.0017	0.002
10	10	0.002	0.002	0.0018	0.0014	0.0013	0.0013	0.0008	0.0006	0.0005	0.0008	0.0007	0.0010	0.0014	0.0018	0.002	0.004
1	25	0.009	0.004	0.003	0.003	0.004	0.0019	0.0013	0.0009	0.0010	0.0014	0.0013	0.0016	0.0015	0.0012	0.003	0.004
2	25	0.008	0.004	0.003	0.002	0.0013	0.0012	0.0008	0.0008	0.0013	0.0010	0.0009	0.0011	0.0012	0.0012	0.002	0.004
3	25	0.006	0.003	0.002	0.002	0.0013	0.0012	0.0011	0.0009	0.0008	0.0010	0.0009	0.0012	0.0015	0.002	0.0018	0.002
4	25	0.004	0.003	0.0018	0.0011	0.0014	0.0011	0.0009	0.0009	0.0006	0.0006	0.0007	0.0011	0.0006	0.0014	0.002	0.003
5	25	0.002	0.002	0.0008	0.0008	0.000	0.0004	0.0005	0.0006	0.0003	0.0005	0.0005	0.0004	0.0005	0.0012	0.0019	0.003
10	25	0.002	0.0013	0.0015	0.0007	0.000	0.0006	0.0004	0.0004	0.0003	0.0004	0.0004	0.0008	0.0009	0.0007	0.0013	0.0010

Appendix 1-11. Radial Standard Deviation, One Shot Group

Table A1-11.1. Percentiles of the Radial Standard Deviation (*RSD*).

n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
2	2.6283	2.3018	2.1460	1.9206	1.7308	1.5174	1.2686	1.1774
3	2.4810	2.2256	2.1037	1.9273	1.7784	1.6103	1.4129	1.3398
4	2.3695	2.1533	2.0501	1.9006	1.7742	1.6313	1.4627	1.4001
5	2.2858	2.0955	2.0045	1.8727	1.7611	1.6347	1.4853	1.4296
6	2.2207	2.0489	1.9668	1.8477	1.7468	1.6323	1.4968	1.4462
7	2.1683	2.0107	1.9353	1.8259	1.7331	1.6279	1.5029	1.4563
8	2.1249	1.9786	1.9086	1.8069	1.7206	1.6227	1.5063	1.4627
9	2.0884	1.9513	1.8856	1.7903	1.7093	1.6173	1.5079	1.4670
10	2.0570	1.9276	1.8656	1.7756	1.6991	1.6121	1.5086	1.4699
11	2.0297	1.9069	1.8480	1.7625	1.6898	1.6071	1.5087	1.4718
12	2.0056	1.8885	1.8323	1.7507	1.6814	1.6024	1.5084	1.4731
13	1.9841	1.8720	1.8183	1.7401	1.6737	1.5980	1.5078	1.4739
14	1.9649	1.8572	1.8056	1.7305	1.6666	1.5938	1.5070	1.4744
15	1.9475	1.8438	1.7940	1.7216	1.6601	1.5899	1.5061	1.4747
16	1.9317	1.8315	1.7835	1.7135	1.6540	1.5862	1.5052	1.4748
17	1.9172	1.8203	1.7738	1.7061	1.6484	1.5827	1.5042	1.4747
18	1.9039	1.8099	1.7648	1.6991	1.6432	1.5794	1.5032	1.4746
19	1.8916	1.8003	1.7565	1.6927	1.6383	1.5763	1.5023	1.4744
20	1.8802	1.7914	1.7487	1.6866	1.6338	1.5734	1.5013	1.4741
21	1.8696	1.7831	1.7415	1.6810	1.6295	1.5706	1.5003	1.4738
22	1.8597	1.7753	1.7348	1.6757	1.6254	1.5680	1.4993	1.4735
23	1.8504	1.7680	1.7284	1.6707	1.6216	1.5655	1.4984	1.4731
24	1.8417	1.7611	1.7224	1.6660	1.6180	1.5631	1.4975	1.4728
25	1.8334	1.7546	1.7168	1.6616	1.6146	1.5609	1.4966	1.4724
26	1.8257	1.7485	1.7114	1.6574	1.6113	1.5587	1.4957	1.4720
27	1.8183	1.7427	1.7064	1.6534	1.6082	1.5566	1.4948	1.4716
28	1.8114	1.7372	1.7016	1.6496	1.6053	1.5546	1.4940	1.4712
29	1.8048	1.7320	1.6970	1.6460	1.6025	1.5527	1.4932	1.4707
30	1.7985	1.7270	1.6926	1.6425	1.5998	1.5509	1.4924	1.4703

Table A1-11.1. Percentiles of the Radial Standard Deviation (*RSD*). (CONTINUED)

n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
2	0.8326	0.5364	0.4724	0.3246	0.2265	0.1591	0.1003	0.0708
3	1.0578	0.8005	0.7413	0.5954	0.4867	0.4018	0.3147	0.2627
4	1.1563	0.9293	0.8761	0.7423	0.6394	0.5562	0.4669	0.4110
5	1.2120	1.0070	0.9585	0.8354	0.7393	0.6603	0.5738	0.5185
6	1.2478	1.0597	1.0148	0.9005	0.8104	0.7356	0.6530	0.5994
7	1.2728	1.0979	1.0561	0.9490	0.8640	0.7932	0.7142	0.6627
8	1.2913	1.1272	1.0878	0.9868	0.9063	0.8388	0.7633	0.7137
9	1.3055	1.1505	1.1132	1.0172	0.9405	0.8761	0.8036	0.7559
10	1.3167	1.1694	1.1339	1.0424	0.9690	0.9072	0.8376	0.7915
11	1.3259	1.1852	1.1512	1.0636	0.9932	0.9337	0.8666	0.8221
12	1.3334	1.1986	1.1660	1.0817	1.0140	0.9567	0.8917	0.8487
13	1.3398	1.2101	1.1787	1.0975	1.0321	0.9767	0.9138	0.8721
14	1.3453	1.2202	1.1898	1.1114	1.0481	0.9944	0.9334	0.8928
15	1.3500	1.2290	1.1997	1.1237	1.0623	1.0102	0.9510	0.9115
16	1.3541	1.2369	1.2084	1.1347	1.0751	1.0244	0.9667	0.9283
17	1.3577	1.2439	1.2163	1.1446	1.0866	1.0373	0.9811	0.9435
18	1.3609	1.2502	1.2233	1.1536	1.0971	1.0490	0.9941	0.9575
19	1.3637	1.2560	1.2298	1.1617	1.1066	1.0597	1.0061	0.9703
20	1.3663	1.2612	1.2357	1.1693	1.1154	1.0695	1.0171	0.9821
21	1.3686	1.2660	1.2411	1.1762	1.1235	1.0786	1.0273	0.9930
22	1.3707	1.2705	1.2460	1.1826	1.1311	1.0871	1.0368	1.0031
23	1.3726	1.2746	1.2506	1.1885	1.1380	1.0949	1.0457	1.0126
24	1.3744	1.2783	1.2549	1.1940	1.1445	1.1023	1.0539	1.0215
25	1.3760	1.2819	1.2589	1.1992	1.1506	1.1091	1.0616	1.0298
26	1.3775	1.2852	1.2626	1.2040	1.1563	1.1156	1.0689	1.0376
27	1.3789	1.2882	1.2661	1.2085	1.1617	1.1216	1.0758	1.0449
28	1.3802	1.2911	1.2694	1.2128	1.1667	1.1274	1.0822	1.0519
29	1.3813	1.2938	1.2725	1.2168	1.1715	1.1328	1.0883	1.0585
30	1.3824	1.2964	1.2754	1.2206	1.1760	1.1379	1.0941	1.0647

Appendix 1-12. RMS of the Radial Standard Deviations, Multiple Shot Groups

Table A1-12.1. Percentiles of the RMS of the Radial Standard Deviations (*RMS-RSD*).

k	n	P 0.999	P 0.995	P 0.99	P 0.975	P 0.95	P 0.9	P 0.8	P 0.75
1	2	2.6283	2.3018	2.1460	1.9206	1.7308	1.5174	1.2686	1.1774
2		2.1487	1.9275	1.8219	1.6691	1.5401	1.3946	1.2236	1.1603
3		1.9347	1.7582	1.6739	1.5518	1.4487	1.3320	1.1943	1.1432
4		1.8071	1.6566	1.5847	1.4805	1.3923	1.2924	1.1742	1.1302
5		1.7201	1.5871	1.5235	1.4312	1.3530	1.2644	1.1594	1.1202
10		1.5052	1.4142	1.3705	1.3071	1.2532	1.1919	1.1189	1.0915
1	3	2.4810	2.2256	2.1037	1.9273	1.7784	1.6103	1.4129	1.3398
2		2.0866	1.9129	1.8299	1.7095	1.6077	1.4923	1.3559	1.3050
3		1.9122	1.7732	1.7068	1.6103	1.5285	1.4356	1.3255	1.2843
4		1.8086	1.6899	1.6330	1.5504	1.4803	1.4006	1.3059	1.2705
5		1.7381	1.6329	1.5825	1.5093	1.4471	1.3763	1.2920	1.2604
10		1.5642	1.4918	1.4571	1.4064	1.3633	1.3141	1.2552	1.2331
1	4	2.3695	2.1533	2.0501	1.9006	1.7742	1.6313	1.4627	1.4001
2		2.0282	1.8808	1.8103	1.7079	1.6212	1.5227	1.4059	1.3622
3		1.8778	1.7597	1.7031	1.6209	1.5511	1.4717	1.3772	1.3418
4		1.7885	1.6874	1.6390	1.5685	1.5086	1.4404	1.3591	1.3286
5		1.7278	1.6382	1.5952	1.5326	1.4794	1.4187	1.3463	1.3191
10		1.5780	1.5162	1.4864	1.4431	1.4061	1.3638	1.3131	1.2940
1	5	2.2858	2.0955	2.0045	1.8727	1.7611	1.6347	1.4853	1.4296
2		1.9812	1.8511	1.7889	1.6984	1.6216	1.5343	1.4306	1.3917
3		1.8471	1.7428	1.6927	1.6200	1.5581	1.4876	1.4036	1.3721
4		1.7676	1.6782	1.6353	1.5729	1.5198	1.4592	1.3868	1.3597
5		1.7135	1.6342	1.5961	1.5407	1.4934	1.4395	1.3750	1.3508
10		1.5801	1.5253	1.4989	1.4603	1.4274	1.3898	1.3447	1.3276
1	10	2.0570	1.9276	1.8656	1.7756	1.6991	1.6121	1.5086	1.4699
2		1.8437	1.7547	1.7120	1.6498	1.5968	1.5364	1.4642	1.4371
3		1.7500	1.6783	1.6439	1.5937	1.5508	1.5019	1.4433	1.4213
4		1.6944	1.6328	1.6032	1.5601	1.5232	1.4811	1.4305	1.4115
5		1.6566	1.6019	1.5755	1.5371	1.5043	1.4667	1.4216	1.4046
10		1.5632	1.5252	1.5068	1.4800	1.4571	1.4307	1.3991	1.3871
1	25	1.8334	1.7546	1.7168	1.6616	1.6146	1.5609	1.4966	1.4724
2		1.7004	1.6458	1.6195	1.5811	1.5484	1.5108	1.4658	1.4488
3		1.6419	1.5977	1.5765	1.5454	1.5188	1.4883	1.4517	1.4378
4		1.6071	1.5691	1.5508	1.5240	1.5011	1.4748	1.4431	1.4312
5		1.5835	1.5497	1.5333	1.5094	1.4890	1.4655	1.4373	1.4266
10		1.5251	1.5014	1.4900	1.4732	1.4589	1.4424	1.4225	1.4149

Table A1-12.1. Percentiles of the RMS of the Radial Standard Deviations (*RMS-RSD*). (CONTINUED)

k	n	P 0.5	P 0.25	P 0.2	P 0.1	P 0.05	P 0.025	P 0.01	P 0.005
1	2	0.8326	0.5364	0.4724	0.3246	0.2265	0.1591	0.1003	0.0708
2		0.9161	0.6933	0.6420	0.5157	0.4215	0.3480	0.2725	0.2275
3		0.9441	0.7588	0.7153	0.6061	0.5221	0.4541	0.3812	0.3356
4		0.9581	0.7961	0.7578	0.6604	0.5844	0.5220	0.4537	0.4099
5		0.9665	0.8208	0.7861	0.6975	0.6277	0.5698	0.5058	0.4643
10		0.9833	0.8790	0.8538	0.7888	0.7366	0.6925	0.6427	0.6097
1	3	1.0578	0.8005	0.7413	0.5954	0.4867	0.4018	0.3147	0.2627
2		1.1064	0.9193	0.8750	0.7626	0.6749	0.6027	0.5238	0.4734
3		1.1225	0.9683	0.9314	0.8369	0.7620	0.6995	0.6299	0.5844
4		1.1306	0.9963	0.9640	0.8809	0.8145	0.7587	0.6960	0.6546
5		1.1354	1.0149	0.9858	0.9108	0.8505	0.7996	0.7421	0.7040
10		1.1451	1.0592	1.0383	0.9840	0.9400	0.9025	0.8595	0.8308
1	4	1.1563	0.9293	0.8761	0.7423	0.6394	0.5562	0.4669	0.4110
2		1.1906	1.0270	0.9879	0.8877	0.8082	0.7419	0.6681	0.6199
3		1.2020	1.0675	1.0351	0.9515	0.8846	0.8282	0.7646	0.7225
4		1.2077	1.0908	1.0625	0.9893	0.9303	0.8804	0.8237	0.7861
5		1.2111	1.1063	1.0808	1.0149	0.9616	0.9163	0.8647	0.8303
10		1.2179	1.1434	1.1252	1.0777	1.0391	1.0060	0.9681	0.9425
1	5	1.2120	1.0070	0.9585	0.8354	0.7393	0.6603	0.5738	0.5185
2		1.2385	1.0914	1.0560	0.9650	0.8923	0.8311	0.7624	0.7171
3		1.2473	1.1266	1.0973	1.0217	0.9608	0.9093	0.8507	0.8118
4		1.2517	1.1468	1.1213	1.0552	1.0018	0.9563	0.9045	0.8699
5		1.2544	1.1603	1.1375	1.0780	1.0297	0.9886	0.9416	0.9101
10		1.2596	1.1928	1.1765	1.1338	1.0990	1.0691	1.0348	1.0117
1	10	1.3167	1.1694	1.1339	1.0424	0.9690	0.9072	0.8376	0.7915
2		1.3292	1.2242	1.1986	1.1323	1.0786	1.0329	0.9806	0.9457
3		1.3334	1.2473	1.2263	1.1717	1.1272	1.0891	1.0455	1.0162
4		1.3354	1.2608	1.2425	1.1949	1.1561	1.1228	1.0846	1.0588
5		1.3367	1.2698	1.2535	1.2107	1.1758	1.1458	1.1113	1.0881
10		1.3392	1.2918	1.2801	1.2496	1.2246	1.2031	1.1782	1.1614
1	25	1.3760	1.2819	1.2589	1.1992	1.1506	1.1091	1.0616	1.0298
2		1.3808	1.3140	1.2976	1.2548	1.2198	1.1898	1.1552	1.1319
3		1.3824	1.3278	1.3143	1.2792	1.2504	1.2257	1.1971	1.1778
4		1.3832	1.3358	1.3242	1.2937	1.2686	1.2471	1.2222	1.2053
5		1.3837	1.3413	1.3308	1.3035	1.2810	1.2617	1.2393	1.2242
10		1.3847	1.3546	1.3472	1.3278	1.3118	1.2980	1.2820	1.2711

Appendix 2. Supplementary Details

Appendix 2-1. Power Curves

A brief treatment of power curves is presented for the examples given in this report. Power curves are plots of the probability of rejection under various alternatives. Consider the null hypothesis (H_0) and the alternate hypothesis (H_a):

$$H_0: \sigma \leq \sigma^*$$

$$H_a: \sigma > \sigma^*.$$

Our desire is to accept product with a population σ less than or equal to σ^* , and reject it otherwise. These hypotheses are evaluated by measuring the dispersion measure, M . Let our test for acceptance be:

$$M \leq M^*.$$

If our measurement M exceeds M^* , then the product is rejected. The probability of rejection is given by:

$$\begin{aligned} P_{rej}(M > M^*) &= P_{rej}(M/\sigma > M^*/\sigma) \\ &= P_{rej}(M/\sigma > (\sigma^*/\sigma)(M^*/\sigma^*)). \end{aligned}$$

For compactness of notation, $T \equiv M/\sigma$. The probability of rejection is now rewritten as:

$$P_{rej}(M > M^*) = P_{rej}(T > (\sigma^*/\sigma)(M^*/\sigma^*)).$$

M^* is chosen such that if H_0 is true then the null hypothesis is rejected at a type-I error rate of no more than $\alpha\%$. This is ensured by the choice:

$$(M^*/\sigma^*) = T_{1-\alpha},$$

where $T_{1-\alpha}$ is the $(1 - \alpha) \times 100$ percentile of T . We now have:

$$P_{rej}(M > M^*) = P_{rej}(T > (\sigma^*/\sigma)T_{1-\alpha}).$$

If H_0 is true, the probability of rejection is:

$$P_{rej}(T > (\sigma^*/\sigma)T_{1-\alpha}) \leq \alpha,$$

where the equality holds for $\sigma = \sigma^*$, and if H_a is true, the probability of rejection is:

$$P_{rej}(T > (\sigma^*/\sigma)T_{1-\alpha}) = 1 - \beta,$$

where β is the type-II error rate, and $(1 - \beta)$ is the power. The Power Curve is constructed by choosing α and plotting the probability of rejection ($P_{rej}(T > (\sigma^*/\sigma)T_{1-\alpha})$) versus the ratio of σ/σ^* .

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List of Abbreviations and Acronyms

AES	Average Extreme Spread
AMD	Average Mean Deviation
AMR	Average Mean Radius
AR	Average Range
ASD	Average Standard Deviation
CEP	Circular Error Probability
CI	Confidence Interval
CV	Coefficient of Variation
CL	Confidence Level
DoF	Degrees of Freedom
EHD	Extreme Horizontal Dispersion
ES	Extreme Spread
EVD	Extreme Vertical Dispersion
HSD	Horizontal Standard Deviation
MCE.....	Monte Carlo Error
MD	Mean Deviation
MHD	Mean Horizontal Deviation
MVD	Mean Vertical Deviation
MR	Mean Radius
R.....	Range
RE	Relative Efficiency
RMS	Root Mean Square
RMS-SD.....	Root Mean Square of Standard Deviations
RMS-RSD	Root Mean Square of Radial Standard Deviations
RSD.....	Radial Standard Deviation
SD	Standard Deviation
SE.....	Standard Error
VSD	Vertical Standard Deviation